PROPERTIES OF SOLUTIONS OF THE KPI EQUATION *

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Abstract

The Kadomtsev–Petviashvili I (KPI) is considered as a useful laboratory for experimenting new theoretical tools able to handle the specific features of integrable models in $2+1$ dimensions. The linearized version of the KPI equation is first considered by solving the initial value problem for different classes of initial data. Properties of the solutions in different cases are analyzed in details. The obtained results are used as a guideline for studying the properties of the solution $u(t,x,y)$ of the Kadomtsev–Petviashvili I (KPI) equation with given initial data $u(0,x,y)$ belonging to the Schwartz space. The spectral theory associated to KPI is studied in the space of the Fourier transform of the solutions. The variables $p=\{p_1,p_2\}$ of the Fourier space are shown to be the most convenient spectral variables to use for spectral data. Spectral data are shown to decay rapidly at large p but to be discontinuous at $p=0$. Direct and inverse problems are solved with special attention to the behaviour of all the quantities involved in the neighborhood of $t=0$ and $p=0$. It is shown in particular that the solution $u(t,x,y)$ has a time derivative discontinuous at $t=0$ and that at any $t\neq 0$ it does not belong to the Schwartz space no matter how small in norm and rapidly decaying at large distances the initial data are chosen.

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1 Introduction

We consider the initial value problem for the Kadomtsev–Petviashvili equation [1, 2, 3] in its version called KPI

$$(u_t - 6uu_x + u_{xxx})_x = 3u_{yy}\,, \qquad u = u(t,x,y)\,,$$
$$u(0,x,y) = u(x,y)\,, \tag{1.1}$$

for $u(t,x,y)$ real. Already in 1974 [4] it has been acknowledged to be integrable since it can be associated to a linear spectral problem and, precisely, to the non–stationary Schrödinger equation

$$(-i\partial_y + \partial_x^2 - u(x,y))\Phi = 0. \tag{1.2}$$

However, to building a complete and coherent theory for the spectral transform of the potential $u(x,y)$ in (1.2) that could be used to linearize the initial value problem of (1.1) resulted to be unexpectedly difficult. The real breakthrough has been the discover that the problem is solvable via a non local Riemann–Hilbert formulation [5, 6].

Successively other progresses have been made. The characterization problem for the spectral data was solved in [7]. The extension of the spectral transform to the case of a potential $u(x,y)$ approaching to zero in every direction except a finite number has been faced in [8]. The questions of the associated conditions (often called 'constraints') and how to choose properly ∂_x^{-1} in the evolution form of KPI

$$u_t(t,x,y) - 6u(t,x,y)u_x(t,x,y) + u_{xxx}(t,x,y) = 3\partial_x^{-1}\,u_{yy}(t,x,y) \tag{1.3}$$

have been studied in [9]. Some relevant points on constraints were known earlier, but have not been published [10]. See also [11]. In the article [12] some rigorous results on the existence and uniqueness of solutions of the direct and inverse problem have been proved.

In this article, as well as in the previous one [13] by the same authors, the description of the KPI equation by means of the inverse spectral transform is revisited. We think that this is necessary since some specific properties of the solutions of the initial value problem (1.1) and, more generally, of the $2+1$–integrable equations are not properly (or in any case not exhaustively) described in the literature. Moreover, these properties are specially interesting since they seem to be peculiar of the integrable equations in $2+1$ dimensions. Our study is essentially based on some new theoretical tools in the theory of the spectral transform developed in [8, 14, 15].

In this article we consider the equation (1.1) for initial data $u(x,y)$ belonging to the Schwartz space $\mathcal{S}$. The main results we get as regards the properties of the solutions and the theory of the spectral transform can be summarized in the following points (the integrations when it is not differently indicated are performed all along the real axis from $-\infty$ to $+\infty$).

1. The solution $u(t,x,y)$ immediately leaves the Schwartz space since at any time $t \neq 0$ develops a tail slowly decreasing for $x \to t\infty$ according to the following

formula

$$u(t,x,y) = \frac{-1}{4\pi\sqrt{3tx}|x|}\iint dx'dy'\, u(x',y') + o\big(|x|^{-3/2}\big)\,. \tag{1.4}$$

Only in the opposite direction, i.e. for $x \to -t\infty$, it decreases rapidly.

2. The problem of the proper choice of ∂_x^{-1} in (1.3) requires a special investigation in the neighborhood of the initial time $t = 0$. The final answer is that, for initial data $u(x,y) \in \mathcal{S}$ which are arbitrarily chosen and, therefore, not necessarily subjected to the constraint

$$\int dx\, u(x,y) = 0\,, \tag{1.5}$$

the function $u(t,x,y)$ reconstructed by solving the inverse spectral problem for (1.2) evolves in time according to the equation

$$u_t(t,x,y) - 6u(t,x,y)u_x(t,x,y) + u_{xxx}(t,x,y) = 3\int_{-t\infty}^{x} dx'\, u_{yy}(t,x',y)\,. \tag{1.6}$$

Notice that the possibility that the inversion of ∂_x can depend on t was already mentioned in the literature for the Davey–Stewartson equation [16].

3. The time derivative $u_t(t,x,y)$ has at $t = 0$ different left and right limits and the condition

$$\int dx\, u(t,x,y) = 0 \tag{1.7}$$

is dynamically generated by the evolution equation at times $t \neq 0$, i.e. for these times we recover the result obtained in [9].

4. The primitives $\int_{\pm\infty}^{x}$ or their antisymmetric combination $\frac{1}{2}\Big(\int_{-\infty}^{x} - \int_{x}^{+\infty}\Big)$ cannot be exchanged with the limit $t \to 0$. In fact we have

$$\begin{aligned} &\lim_{t\to+0}\int_{-\infty}^{x} dx'\, u(t,x',y) = -\lim_{t\to+0}\int_{x}^{+\infty} dx'\, u(t,x',y) = \\ &\lim_{t\to+0}\frac{1}{2}\bigg(\int_{-\infty}^{x} - \int_{x}^{+\infty}\bigg)dx'\, u(t,x',y) = \int_{-\infty}^{x} dx'\, u(x',y) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} &\lim_{t\to-0}\int_{-\infty}^{x} dx'\, u(t,x',y) = -\lim_{t\to-0}\int_{x}^{+\infty} dx'\, u(t,x',y) = \\ &\lim_{t\to-0}\frac{1}{2}\bigg(\int_{-\infty}^{x} - \int_{x}^{+\infty}\bigg)dx'\, u(t,x',y) = -\int_{x}^{+\infty} dx'\, u(x',y)\,. \end{aligned} \tag{1.9}$$

Thus the two forms of writing down the evolution form of the Kadomtsev–Petviashvili equation suggested in [9] and in (1.6) are equivalent but only in the last formulation the integration over x and the limit $t \to 0$ can be exchanged showing explicitly the discontinuity at $t = 0$.

5. Conditions ('constraints') higher than (1.7) can be obtained. But some special care is needed due to the asymptotic behavior of u as given in (1.4). E.g. in the condition

$$\int dx\, x\, u_y(t,x,y) = 0\,, \qquad t \neq 0\,, \tag{1.10}$$

the y-derivative cannot be extracted from the integral. Higher conditions are non-linear and can be given by means of recursion relations.

6. In spite of the dynamic condition (1.7) we have that at all times

$$\int dx \int dy\, u(t,x,y) = \int\int dxdy\, u(0,x,y). \tag{1.11}$$

The integrations in the l.h.s. are performed in the order explicitly indicated. Since no any constraint is imposed to the initial data this quantity is not necessarily equal to zero and gives a nontrivial integral of motion.

7. The properties of the Jost solutions of the equation (1.2) essentially differs at $t = 0$ and $t \neq 0$. In particular, at $t = 0$ the coefficients of the asymptotic $1/\mathbf{k}$–expansion, where $\mathbf{k}$ denoted the spectral parameter, depend on the sign of the imaginary part $\mathbf{k}_{\Im}$ (cf. [10] and [15]), while at $t \neq 0$ these coefficients in the limit $t \to 0$ depend on the sign of t.

8. Spectral data do not belong to the Schwartz space $\mathcal{S}$ and no any small norm or smoothness condition can improve this situation.

All along the article we find convenient to consider the problem in terms of the Fourier transform of u

$$v(t,p) \equiv v(t,p_1,p_2) = \frac{1}{(2\pi)^2}\iint dxdy\, e^{i(p_1x - p_2y)} u(t,x,y)\,, \tag{1.12}$$

where p denotes a 2–component vector

$$p \equiv \{p_1, p_2\}\,. \tag{1.13}$$

Then, the initial value problem (1.1) for the KPI equation can be rewritten as ($d^2q \equiv dq_1dq_2$)

$$\begin{aligned} p_1\frac{\partial v(t,p)}{\partial t} &= -3ip_1^2\int d^2q\, v(t,q)\, v(t,p-q) - i\left(p_1^4 + 3p_2^2\right) v(t,p)\,, \\ v(0,p) &= v(p) \in \mathcal{S}\,, \end{aligned} \tag{1.14}$$

where $v(p)$ is the Fourier transform (1.12) of the initial data $u(x,y)$. The problem of properly defining ∂_x^{-1} in (1.3) is reformulated as the problem of properly regularizing the distribution p_1^{-1}. In terms of the Fourier transform $v(t,p)$ we have instead of (1.6)

$$\frac{\partial v(t,p)}{\partial t} = -3ip_1\int d^2q\, v(t,q)\, v(t,p-q) - i\,\frac{p_1^4 + 3p_2^2}{p_1 + i0t}\, v(t,p) \tag{1.15}$$

and condition (1.7) takes the form

$$\lim_{p_1\to 0} v(t,p_1,p_2)=0, \qquad \text{for} \quad t\neq 0, \tag{1.16}$$

where the limit is understood in the sense of distributions in the p_2 variable.

According to the usual scheme of the spectral transform theory we expect that the spectral data evolve in time as the Fourier transform of the linear part of (1.1). This makes clear that the special behavior in time of the solutions of KPI is just complicated by the nonlinearity of the KPI but is, in fact, inherent to the singular character of its linear part. Due to this we start the article by considering in section 2 the linearized version of the KPI equation. The above mentioned properties are proved for the class of Schwartz initial data. As well some more general classes of initial data are considered, for which the condition of unique solvability of the initial value problem is preserved and the corresponding evolution forms of the equation are presented. The study of the nonlinear case for these initial data is an open problem. It is show that in spite of the discontinuity at $t=0$ the linearized evolution equation for initial data belonging to the Schwartz space is an Hamiltonian system. Thus it is natural to expect that the KPI equation with this type of initial data is Hamiltonian too. But since a formulation of the simplectic and Poisson structures in terms of spectral data meets with some difficulties even in $1+1$ dimensions, i.e. in the KdV case (see, e.g. [17]), we postponed this problem to a forthcoming publication.

In section 3 the inverse spectral transform method for the KPI equation is revisited. We use a specific form of the equations of the inverse problem suggested in [8]. The advantages of this formulation as well as the connections with the results obtained in [13] are given in section 4. A lemma and some propositions on distributions that are of common use in the article are reported in the appendix.

2 The Linearized Equation

We consider the linear part of the KPI equation

$$\partial_x(\partial_t U(t,x,y)+\partial_x^3 U(t,x,y)) = 3\partial_y^2 U(t,x,y). \tag{2.1}$$

and its initial value problem, i.e. we are searching special classes of solutions $U(t,x,y)$ satisfying

$$U(0,x,y)=U(x,y). \tag{2.2}$$

The solution $U(t,x,y)$ and the initial value $U(x,y)$ can be distributions in the (t,x,y) variables belonging to the space $\mathcal{S}'$ dual to the Schwartz space $\mathcal{S}$, but $U(t,x,y)$ must be continuous at $t=0$ in order to have a meaningful initial value problem.

Since (2.1) is not an evolution equation its initial value problem is singular and we must pay special attention to the behaviour of the solution $U(t,x,y)$ at the initial time $t=0$. From the study of this problem we expect to get useful guidelines for the KPI equation. In fact, the additional nonlinear term present in the KPI equation complicates the analysis but does not change the specific behaviour of the solution in the neighborhood to the initial time.

Performing the Fourier transform of the equation (2.1) we get

$$p_1 \frac{\partial V(t,p)}{\partial t} = -i\,(p_1^4 + 3p_2^2)\,V(t,p) \tag{2.3}$$

where

$$V(t,p) \equiv V(t,p_1,p_2) = \frac{1}{(2\pi)^2} \iint dxdy\, e^{i(p_1 x - p_2 y)} U(t,x,y) \tag{2.4}$$

and p denotes a 2–component vector

$$p \equiv \{p_1, p_2\}. \tag{2.5}$$

For any $p_1 \neq 0$ the general solution of (2.3) satisfying the initial value requirement

$$V(0,p) = V(p) \tag{2.6}$$

is given by

$$V(t,p) = \exp\left(-it\,\frac{p_1^4 + 3p_2^2}{p_1}\right) V(p), \tag{2.7}$$

We choose $V(p)$ to be the Fourier transform of the initial data $U(x,y)$

$$V(p) = \frac{1}{(2\pi)^2} \iint dx\,dy\, e^{ip_1 x - ip_2 y} U(x,y). \tag{2.8}$$

Then (2.6) is the Fourier transform of (2.2) and the initial value problem defined in (2.1) and (2.2) has been transformed into an equivalent one in the Fourier transformed space of variables p.

A natural regularization at $p_1 = 0$ of the distribution in (2.7) is just formula (2.7) itself considered for all values of the variables (t,p) including $p_1 = 0$. In order to compute its time derivative it is convenient to consider it as the limit in the sense of $\mathcal{S}'$

$$V(t,p) \equiv \lim_{\epsilon \to +0} \exp\left(-it\frac{p_1^4 + 3p_2^2}{p_1 + i\epsilon t}\right) V(p). \tag{2.9}$$

Then we derive with respect to t both sides and exchange the time derivative and the limit in the r.h.s., which is allowed in the space $\mathcal{S}'$ since a test function multiplied by $\exp\left(-it\frac{p_1^4+3p_2^2}{p_1+i\epsilon t}\right)$ still belongs to the Schwartz space. We get (see lemma A.1 in the appendix for an alternative proof) that $V(t,p)$ satisfies the evolution equation

$$\frac{\partial V(t,p)}{\partial t} = -i\,\frac{p_1^4 + 3p_2^2}{p_1 + i0t}\,V(t,p). \tag{2.10}$$

Note that the special regularization of $1/p_1$ obtained in the r.h.s. is fixed by the requirement that $V(t,p)$ evolves in time according to (2.7) for all values of p_1 including $p_1 = 0$.

The above considerations can be extended to the cases in which $\operatorname{sgn} p_1 \, V(p)$, $p_2 V(p)$ and $\frac{p_2}{|p_1|} V(p)$ belong to $\mathcal{S}$. They need to be considered in details. For instance in the first case the time derivative and the limit cannot be exchanged any more and the time derivative of $V(t,p)$ is not defined at $t = 0$. A complete study of all these cases is considered in the following.

We conclude that $V(t,p)$ furnishes via the Fourier transform

$$U(t,x,y) = \int d^2p e^{-ip_1x+ip_2y} V(t,p) \tag{2.11}$$

a solution $U(t,x,y)$ belonging to the space of distributions $\mathcal{S}'$ satisfying the initial data equation (2.2) and the evolution equation

$$\partial_t U(t,x,y) + \partial_x^3 U(t,x,y) = 3 \int_{-t\infty}^{x} dx' \, \partial_y^2 U(t,x',y). \tag{2.12}$$

Note that this equation must be considered as an equation for distributions in the variables x, y and t. In particular at the initial time $t = 0$ it can be continuous or discontinuous with left and right limits or not defined at all. In the following sections we shall give an explicit example for all these three possibilities.

Since the most general regularization of $V(t,p)$ at $p_1 = 0$ can be obtained by adding a distribution with support the point $p_1 = 0$ (see [18]) we conclude that the general solution of the problem (2.1), (2.2) can be written in the form

$$U(t,x,y) + \Omega(t,x,y), \tag{2.13}$$

where

$$\Omega(t,x,y) = \int d^2p e^{-ip_1x+ip_2y} \omega(t,p) \tag{2.14}$$

is the Fourier transform of a distribution $\omega(t,p)$ concentrated at $p_1 = 0$. We have therefore to find the general solution of the problem

$$p_1 \frac{\partial \omega(t,p)}{\partial t} = -i \, (p_1^4 + 3p_2^2) \, \omega(t,p), \qquad \omega(0,p) = 0, \tag{2.15}$$

where $\omega(t,p)$ is a finite sum of the δ–function $\delta(p_1)$ and its derivatives of the form

$$\omega(t,p) = \sum_{n=0}^{N} \delta^{(n)}(p_1) \, \omega_n(t,p_2), \tag{2.16}$$

with ω_n some distributions in the p_2 and t variables. By inserting (2.16) into (2.15) and by using the known properties

$$p_1 \, \delta(p_1) = 0, \qquad p_1 \, \delta^{(n)}(p_1) = -n\delta^{(n-1)}(p_1), \quad n = 1, 2, \ldots \tag{2.17}$$

we get

$$p_2^2 \omega_N(t,p_2) = 0. \tag{2.18}$$

and, then, by recursion

$$\left(p_2^2\right)^{n+1} \omega_{N-n}(t,p_2) = 0 \qquad n = 1, \ldots, N. \tag{2.19}$$

Using again property (2.17) for p_2 we get that $\omega_n(t,p_2)$ is a sum of the δ-function $\delta(p_2)$ and its derivatives up to the order $2(N-n)+1$ with time dependent coefficients.

This result can be re-expressed in terms of the x and y variables by saying that $\Omega(t,x,y)$ is a polynomial in x and y of the form

$$\Omega(t,x,y) = \sum_{n=0}^{N} \sum_{m=0}^{2(N-n)+1} a_{nm}(t)\, x^n\, y^m. \tag{2.20}$$

The coefficients $a_{nm}(t)$ obey an undetermined linear system of ordinary differential equations, which is obtained by inserting $\Omega(t,x,y)$ into (2.1). It is easy to check (we omit for shortness the corresponding calculations) that all the coefficients $a_{nm}(t)$ for $m = 2, \ldots, 2(N-n)+1$ can be expressed as linear combinations of $a_{n0}(t)$ and $a_{n1}(t)$ and their time derivatives. These functions are arbitrary up to the condition that $\Omega(t,x,y)$ is identically zero at the initial time $t=0$.

We conclude that

Proposition 2.1 *For the initial data $U(x,y)$ we are considering, since they are vanishing at large x (or y), if there exists a solution $U(t,x,y)$ of the linearized KPI equation vanishing at large x (or y) it is unique. This solution is given by the Fourier transform of the distribution $V(t,p)$ defined in (2.7) and satisfies the evolution equation (2.12).*

In the following we will consider four different classes of initial data vanishing at large x or y and we will show that in three cases the Fourier transform of (2.7) satisfies (in the sense of the distributions) (2.12) at all times including $t=0$, vanishes at large x or y and is therefore unique, while in one case the time derivative of the Fourier transform of (2.7) at $t=0$ is not defined and therefore the evolution form of the linearized KP equation has a (unique) solution in the space considered for any $t \neq 0$.

2.1 Some special classes of solutions of the linearized equation

We want to build special classes of solutions of the evolution version of the KPI equation in (2.12). This can be done by noting that, for any distribution $G(t,x,y)$ satisfying the partial differential equation

$$\left(\partial_t \partial_x - 3\partial_y^2\right) G(t,x,y) = 0, \tag{2.21}$$

the convolution

$$\tau(t,x,y) = \int\!\!\int dx' dy' G(t, x-x', y-y') U_0(t,x',y') \tag{2.22}$$

of G with an arbitrary solution U_0 of the equation

$$\left(\partial_t + \partial_x^3\right) U_0(t,x,y) = 0 \tag{2.23}$$

satisfies the linear partial differential equation

$$\left(\partial_t + \partial_x^3\right) \partial_x \tau = 3\partial_y^2 \tau \tag{2.24}$$

and that this last equation can be considered the time evolution version of (2.1) for

$$U(t,x,y) = \partial_x \tau(t,x,y). \tag{2.25}$$

The general solution of (2.23) can be written as

$$U_0(t,x,y) = \iint d^2p\, e^{-ip_1x+ip_2y-itp_1^3} V_0(p) \tag{2.26}$$

with $V_0(p) \in \mathcal{S}'$. We require the following integrals

$$\int dx\, U_0(t,x,y) = \int dx\, U_0(0,x,y) = 2\pi \int dp_2\, e^{ip_2y} V_0(0,p_2) \tag{2.27}$$

$$\int dy\, U_0(t,x,y) = 2\pi \int dp_1\, e^{-ip_1x-itp_1^3} V_0(p_1,0) \tag{2.28}$$

to be well defined and convergent, i.e. we require $V_0(p)$ to be at $p_1 = 0$ and at $p_2 = 0$ a well defined distribution in the variable p_2 and p_1, respectively.

Since we are studying an initial value problem special attention must be paid to the distribution $G(t,x,y)$ and its derivatives at $t = 0$. In particular, since

$$U(0,x,y) = \iint dx'dy'\, \partial_x G(t,x-x',y-y')|_{t=0} U_0(0,x',y') \tag{2.29}$$

the distribution $G(t,x,y)$ must be defined at $t = 0$.

Let us, now, consider the quadratic form

$$Q(t,x,y) = 12xt - y^2 \tag{2.30}$$

and let us define

$$Q_-^\lambda = |Q|^\lambda \vartheta(-Q), \qquad Q_+^\lambda = |Q|^\lambda \vartheta(Q). \tag{2.31}$$

These distributions, following [18], can be defined by analytical continuation in λ. Both have simple poles at $\lambda = -1, -2, \ldots, -k, \ldots$ and Q_+^λ has additional simple poles at $\lambda = -3/2, -3/2-1, \ldots, -3/2-k, \ldots$ The residua at the first kind of poles are distributions with support on the cone surface $Q = 0$ while the residua at the second kind of poles are distributions with support on the origin, i.e. on the vertex of the cone $Q = 0$.

We are interested in using these distributions to build distributions $G(t,x,y)$ satisfying the partial differential equation (2.21). For all the distributions introduced in the following we are using notations and definitions of [18]. For instance the distribution $\frac{1}{x}$ is defined by

$$\int dx\,\frac{1}{x}\varphi(x) \equiv \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} dx\,\frac{1}{x}\varphi(x) \tag{2.32}$$

where $\varphi(x)$ is an arbitrary test function belonging to the Schwartz space $\mathcal{S}$.

We are considering four cases:

Case i)

$$G_+(t,x,y) = \frac{\operatorname{sgn} t}{\pi} Q_+^{-1/2} \tag{2.33}$$

Case ii)

$$\widehat{G}_+^\sigma(t,x,y) = \frac{\operatorname{sgn} t}{\pi}\frac{y}{x} Q_+^{-1/2} - 2\sigma\delta(x)\vartheta(-\sigma y), \qquad \sigma = \pm \tag{2.34}$$

Case iii)

$$G_-(t,x,y) = \frac{1}{\pi} Q_-^{-1/2} \tag{2.35}$$

Case iv)

$$\widehat{G}_-(t,x,y) = \frac{1}{\pi}\frac{y}{x} Q_-^{-1/2}. \tag{2.36}$$

The corresponding solutions will be noted, respectively, $U_+, \widehat{U}_+^\sigma, U_-, \widehat{U}_-$. The initial data $U(0,x,y)$ can be computed in terms of the initial data $U_0(0,x,y)$ of the evolution equation (2.23) that according to the requirements made for the integrals in (2.27) and (2.28) must vanish at large x and y. It results (see in the following) that $U(0,x,y)$ vanishes at large x and does not necessarily vanish at large y. This is sufficient to assure that all four cases furnish solutions belonging to the special class considered in the previous section that admits one and only one solution for the initial value problem.

Case i) It is convenient to exploit the general theorem about tempered distributions assuming that any distribution can be expressed as derivative of a regular function.

It is easy to verify that

$$Q_+^{-1/2} - \pi\delta(y)\vartheta(xt) = \frac{\partial}{\partial y}\left[\left(\arctan\frac{y}{|Q|^{1/2}} - \frac{\pi}{2}\operatorname{sgn} y\right)\vartheta(Q)\right] \tag{2.37}$$

and that

$$Q_+^{-1/2} - \pi\delta(y)\vartheta(xt) = \frac{\partial^2}{\partial y^2}\left[\left(y\arctan\frac{y}{|Q|^{1/2}} - \frac{\pi}{2}|y|\right)\vartheta(Q) + Q_+^{1/2}\right] \tag{2.38}$$

From (2.37) we have

$$G_+(t,x,y)\big|_{t=\pm 0} = \pm\delta(y)\vartheta(\pm x) \tag{2.39}$$

and from (2.38) by deriving with respect to x

$$\frac{\partial}{\partial x}Q_+^{-1/2} - \pi\,\mathrm{sgn}\,t\delta(x)\delta(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\frac{1}{x}Q_+^{1/2}\right] \tag{2.40}$$

and, consequently, at $t=0$

$$\frac{\partial}{\partial x}G_+(t,x,y)\bigg|_{t=\pm 0} = \delta(x)\delta(y). \tag{2.41}$$

From (2.40) by deriving with respect to t we get

$$\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} - 3\frac{\partial^2}{\partial y^2}\right)Q_+^{-1/2} = 2\pi\delta(t)\delta(x)\delta(y). \tag{2.42}$$

From this equation, due to (2.41), it follows that G_+ satisfies (2.21).

We have therefore from (2.25) and (2.41)

$$U_+(0,x,y) = U_0(0,x,y) \tag{2.43}$$

and from (2.22) and (2.39)

$$\lim_{t\to\pm 0}\tau_+(t,x,y) = \int_{\mp\infty}^{x} dx' U_0(0,x',y). \tag{2.44}$$

Case ii) For studying the distribution $\widehat{G}_+^\sigma$ we consider the identity

$$\frac{y}{x}Q_+^{-1/2} + 6\pi t\delta'(y)\vartheta(xt) = -\frac{\partial^2}{\partial y^2}\left[6t\left(\arctan\frac{y}{|Q|^{1/2}} - \frac{\pi}{2}\,\mathrm{sgn}\,y\right)\vartheta(Q) + \frac{1}{2}\frac{y}{x}Q_+^{1/2}\right]. \tag{2.45}$$

From it at $t=0$ we derive

$$\frac{y}{x}Q_+^{-1/2}\bigg|_{t=0} = 0 \tag{2.46}$$

and, consequently,

$$\widehat{G}_+^\sigma(t,x,y)\bigg|_{t=0} = -2\sigma\delta(x)\vartheta(-\sigma y) \tag{2.47}$$

and

$$\frac{\partial}{\partial x}\widehat{G}_+^\sigma(t,x,y)\bigg|_{t=0} = -2\sigma\delta'(x)\vartheta(-\sigma y) \tag{2.48}$$

By deriving (2.45) with respect to x we have

$$\frac{\partial}{\partial x}\left[\frac{y}{x}Q_+^{-1/2}\right] + 6\pi|t|\delta'(y)\delta(x) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\frac{y}{x^2}Q_+^{1/2}\right] \tag{2.49}$$

and, then, by multiplying by $\operatorname{sgn} t$ and by deriving with respect to t

$$\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}-3\frac{\partial^2}{\partial y^2}\right)\left[\operatorname{sgn} t\,\frac{y}{x}Q_+^{-1/2}\right]=-6\pi\delta'(y)\delta(x) \tag{2.50}$$

and at $t=0$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\left[\operatorname{sgn} t\,\frac{y}{x}Q_+^{-1/2}\right]\Big|_{t=0}=-6\pi\delta'(y)\delta(x). \tag{2.51}$$

We deduce that $\widehat{G}_+^\sigma$ satisfies (2.21).

We have therefore

$$\widehat{U}_+^\sigma(0,x,y)=2\int_{\sigma\infty}^{y}dy'\partial_x U_0(0,x,y') \tag{2.52}$$

and

$$\lim_{t\to\pm 0}\widehat{\tau}_+^\sigma(t,x,y)=2\int_{\sigma\infty}^{y}dy' U_0(0,x,y'). \tag{2.53}$$

Note that the initial data satisfy the constraint

$$\int dx\,\widehat{U}_+^\sigma(0,x,y)=0. \tag{2.54}$$

Case iii) Let us, now, consider $Q_-^{-1/2}$. We have

$$Q_-^{-1/2}=-\frac{\partial^2}{\partial y^2}\left[y\ln\frac{\left|y-|Q|^{1/2}\right|}{|12xt|^{1/2}}\vartheta(-Q)+Q_-^{1/2}\right] \tag{2.55}$$

and by deriving it with respect to x

$$\frac{\partial}{\partial x}Q_-^{-1/2}=-\frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\frac{1}{x}Q_-^{1/2}\right] \tag{2.56}$$

and at $t=0$

$$\frac{\partial}{\partial x}Q_-^{-1/2}\Big|_{t=0}=-\frac{1}{x}\delta(y) \tag{2.57}$$

or

$$\frac{\partial}{\partial x}G_-(t,x,y)\Big|_{t=0}=-\frac{1}{\pi x}\delta(y). \tag{2.58}$$

By deriving, in addition, with respect to t we obtain that G_- satisfies (2.21). However by considering (2.55) at $t=0$ it results that G_- is not defined at $t=0$ and, since G_- satisfies (2.21), that also $\partial_t\partial_x G_-$ is not defined at $t=0$.

We have therefore

$$U_-(0,x,y)=-\frac{1}{\pi}\int dx'\frac{1}{x-x'}U_0(0,x',y). \tag{2.59}$$

However, the limit $\tau_-(t,x,y)$ for $t\to\pm 0$ is not defined and the evolution equation satisfied by $U_-(t,x,y)$ has no left and right limits at $t=0$.

Case iv) Let us, now, consider the identity

$$\frac{y}{x}Q_-^{-1/2} = \frac{\partial^2}{\partial y^2}\left[6t\ln\frac{\left|y-|Q|^{1/2}\right|}{|12xt|^{1/2}}\vartheta(-Q)+\frac{1}{2}\frac{y}{x}Q_-^{1/2}\right]. \tag{2.60}$$

From it at $t=0$ we have

$$\widehat{G}_-(t,x,y)\Big|_{t=0} = \frac{1}{\pi}\frac{\operatorname{sgn} y}{x}. \tag{2.61}$$

and

$$\frac{\partial}{\partial x}\widehat{G}_-(t,x,y)\Big|_{t=0} = -\frac{1}{\pi}\frac{\operatorname{sgn} y}{x^2}. \tag{2.62}$$

By deriving (2.60) with respect to t we get

$$\frac{\partial}{\partial t}\left[\frac{y}{x}Q_-^{-1/2}\right] = \frac{\partial^2}{\partial y^2}\left[6\ln\frac{\left|y-|Q|^{1/2}\right|}{|12xt|^{1/2}}\vartheta(-Q)\right] \tag{2.63}$$

and, then, by deriving with respect to x we deduce that $\widehat{G}_-$ satisfies (2.21).

We have therefore

$$\widehat{U}_-(0,x,y) = -\frac{1}{\pi}\left(\int_{-\infty}^{y}+\int_{\infty}^{y}\right)dy'\int dx'\frac{1}{(x-x')^2}U_0(0,x,y) \tag{2.64}$$

and

$$\lim_{t\to\pm 0}\widehat{\tau}_-(t,x,y) = \frac{1}{\pi}\left(\int_{-\infty}^{y}+\int_{\infty}^{y}\right)dy'\int dx'\frac{1}{x-x'}U_0(0,x,y). \tag{2.65}$$

Note that the initial data satisfy the constraint

$$\int dx\,\widehat{U}_-(0,x,y) = 0. \tag{2.66}$$

In all four cases

$$\int dx\,U(t,x,y) = 0 \qquad \text{for } t\neq 0. \tag{2.67}$$

In the cases *ii*) and *iv*) this condition is satisfied also at $t=0$ and, therefore, it can be considered as a constraint imposed to the initial data that is conserved at all times. On the contrary in the cases *i*) and *iii*) the initial data do not satisfy (2.67), which can be considered as a condition on the solution dynamically generated at all times $t\neq 0$ by the evolution equation.

By inserting (2.26) into (2.22) we deduce that τ can be rewritten as

$$\tau(t,x,y) = \int d^2p\, e^{-ip_1x+ip_2y-itp_1^3}H(t,p)V_0(p) \tag{2.68}$$

where $H(t,p)$ is the Fourier transform of $G(t,x,y)$. The computation of $H(t,p)$ allows us to state the connection with the general formula (2.11) given in the previous section and, specifically, to state the behaviour of $V(p)$ at $p_1 = 0$ and at $p_2 = 0$ in the four cases considered via the formula

$$V(t,p) = -ip_1\, e^{-itp_1^3} H(t,p)V_0(p). \tag{2.69}$$

It is convenient to compute $H(t,p)$ as a limit according to the formula

$$H(t,p) = \lim_{\mu\to 0}\lim_{\epsilon_1\to 0}\int dx(x_+^\mu + x_-^\mu)e^{ip_1x-\epsilon_1|x|}\lim_{\epsilon_2\to 0}\int dy e^{-ip_2y-\epsilon_2|y|}G(t,x,y). \tag{2.70}$$

In fact the distribution $(x_+^\mu + x_-^\mu)e^{-\epsilon_1|x|}e^{-\epsilon_2|y|}G(t,x,y)$ converges to $G(t,x,y)$ as $\mu\to 0$, $\epsilon_1\to 0$, $\epsilon_2\to 0$ in the sense of distributions and the Fourier transform operator commute with the limit. The insertion of the term $x_+^\mu + x_-^\mu$ is unessential in the case $G_\pm$ but in the cases $\widehat{G}_\pm$ it takes into account explicitly that we have to compute the principal value of the integration over x.

In all cases we make the change of variables $12x\operatorname{sgn} t \to x$ and $y\to -|xt|^{1/2}y$. For the Bessel functions of different kinds introduced in the following we use the notations of [19].

Case i) For G_+ we get

$$\begin{aligned} H_+(t,p) = {} & \lim_{\epsilon_1\to 0}\frac{\operatorname{sgn} t}{\pi}\int_0^\infty dx e^{i(\frac{p_1}{12}\operatorname{sgn} t+i\epsilon_1)x}\int_{-1}^{+1}dy(1-y^2)^{-1/2}e^{ip_2|xt|^{1/2}y} = \\ & \lim_{\epsilon_1\to 0}\operatorname{sgn} t\int_0^\infty dx e^{i(\frac{p_1}{12}\operatorname{sgn} t+i\epsilon_1)x}J_0\left(p_2|xt|^{1/2}\right) \end{aligned} \tag{2.71}$$

and, therefore,

$$H_+(t,p) = \frac{i}{p_1 + i0t}e^{-3it\frac{p_2^2}{p_1}}. \tag{2.72}$$

Case ii) For $\widehat{G}_+^\sigma$ we get

$$\begin{aligned} & \iint dx\, dy\, e^{ip_1x-ip_2y}\frac{\operatorname{sgn} t}{\pi}\frac{y}{x}Q_+^{-1/2}(t,x,y) = \\ & \lim_{\epsilon_1\to 0} -\frac{|t|^{1/2}}{\pi}\int_0^\infty \frac{dx}{\sqrt{x}}e^{i(\frac{p_1}{12}\operatorname{sgn} t+i\epsilon_1)x}\int_{-1}^{+1}dy\, y(1-y^2)^{-1/2}e^{ip_2|xt|^{1/2}y} = \\ & \lim_{\epsilon_1\to 0} i|t|^{1/2}\int_0^\infty \frac{dx}{\sqrt{x}}e^{i(\frac{p_1}{12}\operatorname{sgn} t+i\epsilon_1)x}J_1(p_2|xt|^{1/2}) = -\frac{2i}{p_2}\left(1-e^{-3it\frac{p_2^2}{p_1}}\right) \end{aligned} \tag{2.73}$$

and, therefore,

$$\widehat{H}_+^\sigma(t,p) = -\frac{2i}{p_2 + i\sigma 0}e^{-3it\frac{p_2^2}{p_1}}. \tag{2.74}$$

Case iii) For G_- we get, analogously,

$$H_-(t,p) = \frac{1}{|p_1|} e^{-3it\frac{p_2^2}{p_1}}. \tag{2.75}$$

Case iv) The computation of $\widehat{H}_-$ is much more involved and the insertion of the term $x_+^\mu + x_-^\mu$ becomes crucial. We obtain

$$\begin{aligned}\widehat{H}_-(t,p) = \lim_{\mu\to 0}\lim_{\epsilon_1\to 0} -\frac{\operatorname{sgn} t}{\pi}|t|^{1/2}\Bigg[\int_0^\infty dx\, x^{\mu-1/2} e^{i(\frac{p_1}{12}\operatorname{sgn} t + i\epsilon_1)x} I^{(-)}(p_2,x,t) - \\ \int_0^\infty dx\, x^{\mu-1/2} e^{-i(\frac{p_1}{12}\operatorname{sgn} t - i\epsilon_1)x} I^{(+)}(p_2,x,t)\Bigg]\end{aligned} \tag{2.76}$$

where

$$\begin{aligned}I^{(-)}(p_2,x,t) = \lim_{\epsilon_2\to 0}\int_1^\infty dy \left(e^{i(p_2+i\epsilon_2)|xt|^{1/2}y} - e^{-i(p_2-i\epsilon_2)|xt|^{1/2}y}\right) y(y^2-1)^{-1/2} = \\ K_1\left(-i(p_2+i0)|xt|^{1/2}\right) - K_1\left(i(p_2-i0)|xt|^{1/2}\right)\end{aligned} \tag{2.77}$$

and

$$\begin{aligned}I^{(+)}(p_2,x,t) = \lim_{\epsilon_2\to 0}\int_0^\infty dy \left(e^{i(p_2+i\epsilon_2)|xt|^{1/2}y} - e^{-i(p_2-i\epsilon_2)|xt|^{1/2}y}\right) y(y^2+1)^{-1/2} = \\ -\frac{i}{2}\Big[K_1\left(-(p_2+i0)|xt|^{1/2}\right) + K_1\left(-(p_2-i0)|xt|^{1/2}\right) - \\ K_1\left((p_2-i0)|xt|^{1/2}\right) - K_1\left((p_2+i0)|xt|^{1/2}\right)\Big].\end{aligned} \tag{2.78}$$

We need, then, to sum six integrals of the form

$$\int_0^\infty dx\, x^{\mu-1/2} e^{-\alpha x} K_1(2\beta\sqrt{x}) = \frac{\Gamma(\mu+1)\Gamma(\mu)}{2\beta}\alpha^{-\mu}\frac{\beta^2}{\alpha}\Psi(\mu+1,2;\frac{\beta^2}{\alpha}) \tag{2.79}$$

extended to complex values of α and β. Note that (2.79) at $\mu = 0$, due to the presence of the gamma function $\Gamma(\mu)$, has a simple pole which must cancel out by summing up all terms in (2.76).

In order to perform correctly the analytical continuation in α and β and in order to extract the singular term at $\mu = 0$ it is convenient to rewrite the confluent hypergeometric function of second kind Ψ in terms of the confluent hypergeometric function Φ which is an entire function by using the formula (see (63) at page 261 in [19])

$$\begin{aligned}\Gamma(\mu)\Gamma(\mu+1)\Psi(\mu+1,2;z) = \Gamma(\mu+1)\Phi(\mu+1,2;z)\log z + \\ \Gamma(\mu+1)\sum_{n=0}^\infty \frac{(\mu+1)_n}{(2)_n}\left[\psi(\mu+1+n) - \psi(1+n) - \psi(2+n)\right]\frac{z^n}{n!} + \Gamma(\mu)\frac{1}{z}.\end{aligned} \tag{2.80}$$

Therefore we get, for complex values of α and β,

$$
\begin{aligned}
&\int_0^\infty dx\, x^{\mu-1/2} e^{-\alpha x} K_1(2\beta\sqrt{x}) = \\
&\frac{\Gamma(\mu+1)}{2\beta}\alpha^{-\mu}\frac{\beta^2}{\alpha}\Phi(\mu+1,2;\frac{\beta^2}{\alpha})[2\log\beta - \log\alpha] + \\
&\frac{\Gamma(\mu+1)}{2\beta}\alpha^{-\mu}\frac{\beta^2}{\alpha}\sum_{n=0}^{\infty}\frac{(\mu+1)_n}{(2)_n}\left[\psi(\mu+1+n)-\psi(1+n)-\psi(2+n)\right]\frac{\left(\frac{\beta^2}{\alpha}\right)^n}{n!} + \\
&\frac{\Gamma(\mu+1)}{2\beta}\alpha^{-\mu}.
\end{aligned}
\tag{2.81}
$$

By inserting it into the right hand side of (2.76) we verify that the coefficient of $\Gamma(\mu)$ at $\mu = 0$ is zero. Therefore the limit $\mu \to 0$ can be computed by using the Hospital rule. It is convenient to consider first $\widehat{H}_-(t,p)$ for $p_1 \neq 0$. By recalling that

$$
\Phi(1,2;z) = \frac{1}{z}(e^z - 1) \tag{2.82}
$$

we get ($p_1 \neq 0$)

$$
\widehat{H}_-(t,p) = \frac{2}{p_2}\,\mathrm{sgn}\, p_1 e^{-3it\frac{p_2^2}{p_1}}. \tag{2.83}
$$

The value of $\widehat{H}_-(t,p)$ at $p_1 = 0$ must be computed separately and we get

$$
\widehat{H}_-(t,p)\Big|_{p_1=0} = 0. \tag{2.84}
$$

We get, therefore, for τ

Case i)

$$
\tau_+(t,x,y) = \int d^2p\, e^{-ip_1x+ip_2y-itp_1^3-3it\frac{p_2^2}{p_1}}\frac{i}{p_1+i0t}V_0(p) \tag{2.85}
$$

Case ii)

$$
\widehat{\tau}_+(t,x,y) = -\int d^2p\, e^{-ip_1x+ip_2y-itp_1^3-3it\frac{p_2^2}{p_1}}\frac{2i}{p_2+i\sigma 0}V_0(p) \tag{2.86}
$$

Case iii)

$$
\tau_-(t,x,y) = \int d^2p\, e^{-ip_1x+ip_2y-itp_1^3-3it\frac{p_2^2}{p_1}}\frac{1}{|p_1|}V_0(p) \qquad (t \neq 0) \tag{2.87}
$$

Case iv)

$$\widehat{\tau}_-(t,x,y) = \int d^2p\, e^{-ip_1x+ip_2y-itp_1^3-3it\frac{p_2^2}{p_1}}\, \frac{2\operatorname{sgn}p_1}{p_2} V_0(p). \tag{2.88}$$

We are, now, able to give more specific information on the general distributional formula

$$\tau(t,x,y) = \int_{-t\infty}^{x} dx'\, \partial_y^2 U(t,x',y) \tag{2.89}$$

derived in the previous section. In the cases $ii)$ and $iv)$, since the condition (2.67) is satisfied at all times including $t=0$, any other operator ∂_x^{-1} can be chosen instead of $\int_{-t\infty}^{x} dx$, and τ and, consequently, $\partial_t U(t,x,y)$ have a definite limit at $t=0$. In the case $i)$ τ has definite right and left limit at $t=0$ and $\partial_t U(t,x,y)$ is in general discontinuous at $t=0$. In the case $iii)$ τ is not defined at $t=0$ and at this point the distribution $\partial_t U(t,x,y)$ is not regular.

For V we get

Case i)

$$V_+(t,p) = e^{-itp_1^3-3it\frac{p_2^2}{p_1}}\, V_0(p) \tag{2.90}$$

Case ii)

$$\widehat{V}_+^{\sigma}(t,p) = -e^{-itp_1^3-3it\frac{p_2^2}{p_1}}\, \frac{2p_1}{p_2+i\sigma 0} V_0(p) \tag{2.91}$$

Case iii)

$$V_-(t,p) = -i\operatorname{sgn}p_1\, e^{-itp_1^3-3it\frac{p_2^2}{p_1}}\, V_0(p) \tag{2.92}$$

Case iv)

$$\widehat{V}_-(t,p) = -i\, e^{-itp_1^3-3it\frac{p_2^2}{p_1}}\, \frac{2|p_1|}{p_2} V_0(p). \tag{2.93}$$

By taking into account that $V_0(p)$ is supposed to be continuous at $p_1=0$ and at $p_2=0$ the equations above show explicitly the behaviour of $V(t,p)$ at these two special points in the four considered cases.

2.2 More on the case of smooth initial data

We consider here, in more details, the case of initial data $U(x,y)$ (and then $V(p)$) belonging to the test function Schwartz space $\mathcal{S}$ in the x and y (correspondingly p) variables, i.e. the case $i)$ of the previous section.

Let us recall that in this case the solution $U(t,x,y)$ of the linearized KPI equation satisfies the evolution equation

$$\partial_t U(t,x,y) = -\partial_x^3 U(t,x,y) + 3\partial_y^2 \int_{-t\infty}^{x} dx' U(t,x',y) \tag{2.94}$$

and can be written as

$$U(t,x,y) = \iint dx' dy'\, J(t,x-x',y-y')\, U_0(t,x',y') \tag{2.95}$$

where we have introduced for convenience the notation

$$J(t,x,y) \equiv \partial_x G_+(t,x,y) = \frac{\operatorname{sgn} t}{\pi} \partial_x (12xt - y^2)_+^{-1/2}. \tag{2.96}$$

The initial value of the evolution equation

$$\partial_t U_0(t,x,y) = -\partial_x^3 U_0(t,x,y) \tag{2.97}$$

satisfied by $U_0(t,x,y)$ coincides with the initial value of $U(t,x,y)$, i.e.

$$U_0(0,x,y) = U(x,y). \tag{2.98}$$

Since, as we proved in the previous section,

$$J(t,x,y) = \iint d^2p \, \exp\left(-ixp_1 + iyp_2 - 3it\, \frac{p_2^2}{p_1}\right) \tag{2.99}$$

the solution can also be rewritten as

$$U(t,x,y) = \int d^2p\, e^{-ip_1x + ip_2y - itp_1^3 - 3it\frac{p_2^2}{p_1}} V(p) \tag{2.100}$$

where

$$V(p) = \frac{1}{(2\pi)^2} \iint dx\, dy\, e^{ip_1x - ip_2y} U(x,y). \tag{2.101}$$

The last formula for $U(t,x,y)$ can be used to get an integral of motion. In fact by integrating (2.100) first in y and then in x we obtain that

$$\int dx \int dy\, U(t,x,y) = V(0) = \iint dxdy\, U(x,y) \tag{2.102}$$

for any t. This result does not contradict the condition

$$\int dx\, U(t,x,y) = 0, \qquad t \neq 0, \tag{2.103}$$

that we proved, in the previous section, to be satisfied at any time $t \neq 0$ since the order of integration in (2.102) cannot be exchanged.

By means of (2.95) and (2.96) we derive that $U(t, x, y)$, for t ($t \neq 0$) and y fixed, decreases rapidly for $x \to -t\infty$ while for $x \to t\infty$ it satisfies the following asymptotic behavior

$$\begin{aligned} U(t,x,y) = {} & \frac{-1}{4\pi\sqrt{3tx}|x|}\iint dx'dy'\, U(x',y') \\ & -\frac{1}{32\pi|t|\sqrt{3tx}x^2}\iint dx'dy'\, [12tx' + (y-y')^2]\, U(x',y') + o(|x|^{-5/2}). \end{aligned} \tag{2.104}$$

This asymptotic expansion is differentiable in t and twice differentiable in y and obeys (2.1). Therefore we can differentiate in t the condition (2.103) and use (2.94). This procedure leads to the next condition

$$\int dx\, x\, U_{yy}(t,x,y) = 0, \qquad t \neq 0. \tag{2.105}$$

Thanks to the asymptotic behavior of $U(t, x, y)$ at large x derived in (2.104) one y-derivative can be extracted from the integral getting

$$\int dx\, x\, U_y(t,x,y) = 0, \qquad t \neq 0, \tag{2.106}$$

but it is impossible to remove the second derivative since $\int dx\, x\, U(t,x,y)$ is divergent. This procedure can be continued to get an infinite set of dynamically generated conditions.

Equation (2.104) explains the role of constraints, i.e. conditions of the type (2.103) and (2.106) imposed on the initial data $U(x, y)$, in the asymptotic behavior of $U(t, x, y)$ at large x. Precisely, for each additional constraint that is satisfied we get an additional x^{-1} factor in the decreasing law.

We want, now, to prove that the dynamical system described by the evolution equation (2.94) is Hamiltonian.

We define the Poisson brackets of two functionals F and G which are differentiable with respect to the smooth initial data $U(x, y)$ as follows

$$\{F,\, G\} = \iint d\xi d\eta\, \frac{\delta F}{\delta U(\xi,\eta)} \frac{\partial}{\partial \xi} \frac{\delta G}{\delta U(\xi,\eta)}, \tag{2.107}$$

Then

$$\{U(x,y),\, U(x',y')\} = \partial_x \delta(x-x')\, \delta(y-y') \tag{2.108}$$

and

$$\{V(p),\, V(p')\} = \frac{-ip_1}{(2\pi)^2}\, \delta(p_1+p_1')\, \delta(p_2+p_2'). \tag{2.109}$$

Due to (2.97) and (2.98) we easily find that, at equal times, also $U_0(t,x,y)$ are canonical since they satisfy the same Poisson brackets

$$\{U_0(t,x,y),\, U_0(t,x',y')\} = \partial_x\delta(x-x')\,\delta(y-y'). \tag{2.110}$$

By considering $U(t,x,y)$ as given by (2.100) and by using (2.109) we have

$$\begin{aligned}
&\{U(t,x,y),\, U(t,x',y')\} = \\
&\quad \int d^2p \int d^2p'\, e^{-ip_1x+ip_2y-itp_1^3-3it\frac{p_2^2}{p_1}}\, e^{-ip'_1x'+ip'_2y'-itp'^3_1-3it\frac{p'^2_2}{p'_1}} \times \\
&\quad \frac{-ip_1}{(2\pi)^2}\,\delta(p_1+p'_1)\,\delta(p_2+p'_2) = \\
&\quad \partial_x\delta(x-x')\,\delta(y-y').
\end{aligned} \tag{2.111}$$

Let us introduce the Hamiltonian

$$H = \frac{1}{2}\iint dxdy\,(\partial_x U(t,x,y))^2 - \frac{3}{4}\int dx' \mid x' \mid \iint dxdy\big(\partial_y U(t,x,y)\big)\big(\partial_y U(t,x-x',y)\big). \tag{2.112}$$

Notice that the integration in x' in the second term cannot be performed before the integration in x and y since the corresponding integral does not exist.

We prove the following

Proposition 2.2

i) The Hamiltonian H is an integral of motion.

ii) The time evolution equation (2.94) is Hamiltonian, i.e. it can be rewritten as follows

$$\partial_t U(t,x,y) = \{U(t,x,y), H\}. \tag{2.113}$$

iii) The Poisson structure introduced in (2.107) is not continuous in t at $t=0$

$$\lim_{t\to\pm 0}\{U(t,x,y),H\} \neq \{U(0,x,y),H\} \tag{2.114}$$

Proof.

i) Due to (2.100) definition (2.112) can be rewritten as

$$H = 2\pi^2\int d^2p\,\left(p_1^2+3p_2^2\,p_1^{-2}\right)V(p)\,V(-p), \tag{2.115}$$

which is explicitly time independent. Notice that p_1^{-2} in this expression, which has been obtained as the Fourier transform of $|x'|$, is a distribution defined as follows (see [18])

$$\int dp_1\, p_1^{-2}\,\varphi(p_1) \equiv \lim_{\epsilon\to 0}\int_{|p|\geq\epsilon} dp_1\,\frac{\varphi(p_1)-\varphi(0)}{p_1^2}, \tag{2.116}$$

where $\varphi(p_1)$ is an arbitrary test function belonging to the Schwartz space $\mathcal{S}$.

ii) We use (2.109) and (2.115) to write down

$$\{V(p), H\} = -i\,\frac{p_1^4 + 3p_2^2}{p_1}\,V(p), \tag{2.117}$$

where $1/p_1$ has to be understood in the sense of the principal value. Now from (2.7), by noting that due to the presence of the exponential term for $t \neq 0$

$$\lim_{p_1 \to 0} V(t, p_1, p_2) = 0 \tag{2.118}$$

in the sense of distributions in the variable p_2, we get the evolution equation (2.10) for any $t \neq 0$.

iii) Considering the limit for $t \to \pm 0$ we get by using the lemma A.1 in the appendix

$$\lim_{t\to\pm 0}\{U(t,x,y), H\} = -\partial_x^3 U(0,x,y) + 3\int_{\mp\infty}^{x} dx'\,\partial_y^2 U(0,x',y). \tag{2.119}$$

On the other side due to (2.8)

$$\begin{aligned}\{U(0,x,y), H\} = \{U(x,y), H\} = \\ \int d^2p\, e^{-ip_1x+ip_2y}\,\{V(p), H\} = \\ -\partial_x^3 U(0,x,y) + \frac{3}{2}\int dx'\,\mathrm{sgn}(x-x')\partial_y^2 U(0,x',y),\end{aligned} \tag{2.120}$$

where in the last line we used (2.117).

3 Nonlinear case

3.1 Initial remarks and notations

The analysis of the linearized version of the KPI equation, as performed in the previous section, indicates clearly that it is convenient to study the KPI equation in the space $p = \{p_1, p_2\}$ of the Fourier transform of the solutions

$$v(t,p) \equiv v(t,p_1,p_2) = \frac{1}{(2\pi)^2}\iint dx dy\, e^{i(p_1x - p_2y)} u(t,x,y). \tag{3.1}$$

Correspondingly, also the nonstationary Schrödinger equation

$$(-i\partial_y + \partial_x^2 - u(x,y))\Phi = 0, \tag{3.2}$$

which is the spectral equation associated to KPI, has to be stated in the Fourier space of the variables $p = \{p_1, p_2\}$.

If $\Phi(x,y,\mathbf{k})$ is the Jost solution of (3.2) the corresponding Jost solution in the $p-$space can be defined by first shifting $\Phi(x,y,\mathbf{k})$ and then by taking the Fourier transform according to the following formula

$$\nu(p|\mathbf{k}) = \frac{1}{(2\pi)^2}\int dx\, dy\, e^{ip_1x-ip_2y}e^{i\mathbf{k}x-i\mathbf{k}^2y}\Phi(x,y,\mathbf{k})\,. \tag{3.3}$$

The specific role played by the spectral parameter $\mathbf{k}$ is emphasized by separating it from the p variable by a $|$ and a **bold** font is used to indicate that $\mathbf{k}$ belongs to the complex plane.

Then the spectral equation (3.2) takes the form

$$\big[\mathcal{L}(p) - 2p_1\mathbf{k}\big]\nu(p|\mathbf{k}) = \int d^2p'\, v(p-p')\,\nu(p'|\mathbf{k})\,, \tag{3.4}$$

where

$$\mathcal{L}(p) = p_2 - p_1^2\,. \tag{3.5}$$

The integral equation determining the Jost solution [5] has, in the Fourier space, the form

$$\nu(p|\mathbf{k}) = \delta^2(p) + \frac{\rho(p|\mathbf{k})}{\mathcal{L}(p) - 2p_1\mathbf{k}}\,, \tag{3.6}$$

where

$$\rho(p|\mathbf{k}) = \int d^2p'\, v(p-p')\,\nu(p'|\mathbf{k}) \tag{3.7}$$

and $\delta^2(p) \equiv \delta(p_1)\delta(p_2)$. The new function

$$\rho(p|\mathbf{k}) = [\mathcal{L}(p) - 2p_1\mathbf{k}]\,\nu(p|\mathbf{k}) \tag{3.8}$$

will play a crucial role in the following. Let us introduce the boundary values of ν and ρ on the real axis of the $\mathbf{k}$–plane

$$\nu^{\pm}(p|k) = \lim_{\mathbf{k}_{\Im}\to\pm 0}\nu(p|\mathbf{k}), \qquad \rho^{\pm}(p|k) = \lim_{\mathbf{k}_{\Im}\to\pm 0}\rho(p|\mathbf{k}), \qquad k = \mathbf{k}_{\Re} \tag{3.9}$$

and write

$$\nu^{\pm}(p|k) = \delta^2(p) + \frac{\rho^{\pm}(p|k)}{\mathcal{L}(p) - 2p_1k \mp i0p_1}. \tag{3.10}$$

Then the spectral data are most naturally obtained (see [15]) by computing $\rho(p|k)$ at the special value of k that vanishes the denominator

$$\rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = \int d^2p'\, v(p-p')\,\nu^{\pm}\left(p'\Big|\frac{\mathcal{L}(p)}{2p_1}\right)\,. \tag{3.11}$$

We conclude that the variables $p=\{p_1,p_2\}$ can be considered as the spectral variables of the spectral data. For their introduction in the general scheme of the resolvent approach see [15, 14, 8] and for analogous variables used in the KPII case see [20]. They are related to the more familiar spectral variables $\{\alpha, \beta\}$ (see e.g. [5, 6, 7]) by the formulae

$$\begin{aligned} \alpha &= \frac{\mathcal{L}(-p)}{-2p_1}\,, \qquad & \beta &= \frac{\mathcal{L}(p)}{2p_1} \\ p_1 &= \alpha - \beta\,, & p_2 &= \alpha^2 - \beta^2\,. \end{aligned} \tag{3.12}$$

We denote the spectral data in the variables $\{\alpha, \beta\}$ by $r^{\pm}(\alpha, \beta)$. They are related to the previous ones by

$$r^{\pm}(\alpha,\beta) = -2\pi i \rho^{\pm}(\beta-\alpha, \beta^2-\alpha^2|\alpha)\,, \qquad \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = \frac{i}{2\pi} r^{\pm}\left(\frac{\mathcal{L}(p)}{2p_1}, \frac{\mathcal{L}(-p)}{-2p_1}\right). \tag{3.13}$$

Spectral data using the standard spectral variables $\{\alpha, \beta\}$ evolve in time according to the formula [7]

$$r^{\pm}(t,\alpha,\beta) = \exp\left(4it(\alpha^3-\beta^3)\right)\, r^{\pm}(\alpha,\beta)\,, \tag{3.14}$$

while the spectral data $\rho^{\pm}$ evolve in time as follows

$$\rho^{\pm}\left(t, p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = \exp\left(-it\,\frac{p_1^4+3p_2^2}{p_1}\right)\, \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right), \tag{3.15}$$

i.e. in the same way as the linearized KPI equation in the Fourier transformed space. In fact this is the deep reason of the privileged role played by the $\{p_1, p_2\}$ variables in all the theory.

Note that the Jacobian of the transformation $\{\alpha, \beta\} \to \{p_1, p_2\}$

$$\frac{\partial(p_1,p_2)}{\partial(\alpha,\beta)} = 2|\alpha-\beta| \tag{3.16}$$

is not bounded and not always different from zero. Therefore, the use of the two sets of spectral variables can be not equivalent. In fact (see [13]) the use of the $\{\alpha, \beta\}$ variables requires to perform a subtraction in some formulae of the direct and inverse problem, while, as we will show in the following, this is no more necessary if one uses the $\{p_1, p_2\}$ variables.

Let $\boldsymbol{\mathcal{R}}^{\sigma}_{\pm}$ denotes the integral operator with kernel

$$\mathcal{R}^{\sigma}_{\pm}(\alpha,\beta) = \delta(\alpha-\beta) \mp \theta(\mp\sigma(\alpha-\beta))\, r^{\sigma}(\alpha,\beta)\,, \qquad \sigma = +,-\,. \tag{3.17}$$

Then the characterization conditions for the spectral data [7] can be written as (cf. [15])

$$\boldsymbol{\mathcal{R}}^{\sigma}_{\pm}\, \boldsymbol{\mathcal{R}}^{-\sigma\,\dagger}_{\pm} = \boldsymbol{\mathcal{I}}, \qquad \sigma = +,-\,, \tag{3.18}$$

$$\boldsymbol{\mathcal{R}}^{\sigma}_{+}\, \boldsymbol{\mathcal{R}}^{\sigma\,\dagger}_{+} = \boldsymbol{\mathcal{R}}^{\sigma}_{-}\, \boldsymbol{\mathcal{R}}^{\sigma\,\dagger}_{-}\,. \tag{3.19}$$

If the integral equation in (3.6), (3.7) has no homogeneous solutions, which we always suppose, the operator $\boldsymbol{\mathcal{R}}^{\sigma}_{\pm}$ satisfies also the equation

$$\boldsymbol{\mathcal{R}}^{-\sigma\,\dagger}_{\pm}\,\boldsymbol{\mathcal{R}}^{\sigma}_{\pm} = \boldsymbol{\mathcal{I}}\,, \qquad \sigma = +,-\,, \tag{3.20}$$

which is complementary to (3.18). Two other sets of spectral data are used in literature, the so called unitary $\boldsymbol{\mathcal{S}}$-matrix

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{R}}^{\sigma\,\dagger}_{+}\,\boldsymbol{\mathcal{R}}^{-\sigma}_{-}\,, \qquad \boldsymbol{\mathcal{S}}\,\boldsymbol{\mathcal{S}}^{\dagger} = \boldsymbol{\mathcal{I}} = \boldsymbol{\mathcal{S}}^{\dagger}\,\boldsymbol{\mathcal{S}}\,, \qquad \sigma = +,-, \tag{3.21}$$

and the self-adjoint integral operator $\boldsymbol{\mathcal{F}}^{\sigma}$

$$\boldsymbol{\mathcal{F}}^{-\sigma} = \boldsymbol{\mathcal{R}}^{\sigma}_{\pm}\,\boldsymbol{\mathcal{R}}^{\sigma\,\dagger}_{\pm}\,, \qquad \boldsymbol{\mathcal{F}}^{\sigma\,\dagger} = \boldsymbol{\mathcal{F}}^{\sigma}\,, \qquad \sigma = +,-\,, \qquad \boldsymbol{\mathcal{F}}^{\pm}\,\boldsymbol{\mathcal{F}}^{\mp} = \boldsymbol{\mathcal{I}}. \tag{3.22}$$

Notice that, thanks to (3.19), the operator $\boldsymbol{\mathcal{F}}^{\sigma}$ as defined in (3.22) is independent on the signs $\pm$ of the constituent operators $\boldsymbol{\mathcal{R}}^{\sigma}_{\pm}$. Multiplying (3.19) by $\boldsymbol{\mathcal{R}}^{-\sigma\,\dagger}_{+}$ from the left and by $\boldsymbol{\mathcal{R}}^{-\sigma}_{-}$ from the right we get $\boldsymbol{\mathcal{R}}^{\sigma\,\dagger}_{+}\boldsymbol{\mathcal{R}}^{-\sigma}_{-} = \boldsymbol{\mathcal{R}}^{-\sigma\,\dagger}_{+}\boldsymbol{\mathcal{R}}^{\sigma}_{-}$, due to (3.20). This proves the independence of the $\boldsymbol{\mathcal{S}}$-matrix defined in (3.21) on the sign σ. Properties of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{F}}^{\sigma}$ given in (3.21) and (3.22) follow as well from (3.18) and (3.20). Taking into account the triangularity property of the operators $\boldsymbol{\mathcal{R}}^{\sigma}_{\pm}$ the six equations (3.18) and (3.19) reduce to the four ones [7]

$$r^{\sigma}(\alpha,\beta) + \overline{r^{-\sigma}}(\beta,\alpha) = \sigma\int_{\alpha}^{\beta} d\gamma\, r^{\sigma}(\alpha,\gamma)\,\overline{r^{-\sigma}}(\beta,\gamma) \tag{3.23}$$

$$r^{\sigma}(\alpha,\beta) + \overline{r^{\sigma}}(\beta,\alpha) = \sigma\left(\int_{\alpha}^{\infty} - \int_{-\infty}^{\beta}\right) d\gamma\, r^{\sigma}(\alpha,\gamma)\,\overline{r^{\sigma}}(\beta,\gamma) \tag{3.24}$$

and, analogously, instead of (3.20) we can write

$$r^{\sigma}(\alpha,\beta) + \overline{r^{-\sigma}}(\beta,\alpha) = \sigma\int_{\alpha}^{\beta} d\gamma\, r^{\sigma}(\gamma,\beta)\,\overline{r^{-\sigma}}(\gamma,\alpha)\,. \tag{3.25}$$

Using now (3.13) we can rewrite (3.23) and (3.24) in terms of ρ

$$\begin{aligned}
&\rho^{\sigma}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) - \overline{\rho^{\sigma'}}\left(-p\Big|\frac{\mathcal{L}(-p)}{-2p_1}\right) = \\
&\quad 2i\pi\sigma p_1\int d^2p'\ \mathrm{sgn}(p_1p'_1)\vartheta\big(-\sigma\sigma'(p_1-p'_1)p'_1\big)\,\delta\big(p_1\mathcal{L}(p') - p'_1\mathcal{L}(p)\big)\times \\
&\qquad \rho^{\sigma}\left(p'\Big|\frac{\mathcal{L}(p')}{2p'_1}\right)\overline{\rho^{\sigma'}}\left(p'-p\Big|\frac{\mathcal{L}(p'-p)}{2(p'_1-p_1)}\right)
\end{aligned} \tag{3.26}$$

Notice that due to the δ–function and (3.5) in the integrand on the r.h.s. we have

$$\frac{\mathcal{L}(p')}{2p'_1} = \frac{\mathcal{L}(p)}{2p_1}\,, \qquad \frac{\mathcal{L}(p'-p)}{2(p'_1-p_1)} = \frac{\mathcal{L}(-p)}{-2p_1}\,. \tag{3.27}$$

Correspondingly from (3.25) we have

$$\rho^{\sigma}\left(p\middle|\frac{\mathcal{L}(p)}{2p_1}\right) - \overline{\rho^{-\sigma}}\left(-p\middle|\frac{\mathcal{L}(-p)}{-2p_1}\right) =$$
$$2i\pi\sigma p_1\int d^2p'\,\vartheta\big((p_1-p_1')p_1'\big)\,\delta\big((p_1-p_1')\mathcal{L}(p)-p_1\mathcal{L}(p-p')\big)\times$$
$$\rho^{\sigma}\left(p'\middle|\frac{\mathcal{L}(p')}{2p_1'}\right)\overline{\rho^{-\sigma}}\left(p-p'\middle|\frac{\mathcal{L}(p-p')}{2(p_1-p_1')}\right). \tag{3.28}$$

3.2 Properties of Jost solutions and Spectral Data

3.2.1 1/k–expansion of the Jost solution at $t = 0$

To get the asymptotic $1/\mathbf{k}$–expansion of the Jost solution we consider the expansion of the kernel $(\mathcal{L}(p) - 2p_1\mathbf{k})^{-1}$ of the integral equation (3.6). By considering it as a distribution in the variables $\mathcal{L}(p)$ and p_1 we have that

$$\frac{1}{\mathcal{L}(p)-2p_1\mathbf{k}} = -\sum_{n=0}^{\infty}\frac{\mathcal{L}(p)^n}{(2\mathbf{k})^{n+1}(p_1\pm i0\mathcal{L}(p))^{n+1}}\,,\quad \mathbf{k}\to\infty,\quad \pm\,\mathbf{k}_{\Im}>0. \tag{3.29}$$

Note that the coefficients of the expansion depend on the half-plane in which $\mathbf{k}$ tends to infinity. Consequently, also the expansion of the Jost solution at large $\mathbf{k}$ depends on the half plane in which the limit is performed. If we note the coefficients of the asymptotic expansion as follows

$$\nu(p|\mathbf{k}) = \sum_{n=0}^{\infty}\frac{\nu_n^{\pm}(p)}{(2\mathbf{k})^n},\quad \rho(p|\mathbf{k}) = \sum_{n=0}^{\infty}\frac{\rho_n^{\pm}(p)}{(2\mathbf{k})^n},\quad \mathbf{k}\to\infty,\quad \pm\,\mathbf{k}_{\Im}>0 \tag{3.30}$$

we have from (3.6) that

$$\nu_0^{\pm}(p) = \delta^2(p)\,,\qquad \rho_0^{\pm}(p) = v(p)\,,$$
$$\nu_n^{\pm}(p) = -\sum_{m=0}^{n-1}\frac{\mathcal{L}(p)^{n-m-1}\,\rho_m^{\pm}(p)}{(p_1\pm i0\mathcal{L}(p))^{n-m}}\,,\qquad n=1,2,\ldots, \tag{3.31}$$

where $\rho_n^{\pm}(p)$ due to the (3.7) and (3.29) obey the recursion relation

$$\rho_n^{\pm}(p) = -\sum_{m=0}^{n-1}\int\frac{d^2q\,v(p-q)\,\mathcal{L}(q)^{n-m-1}}{(q_1\pm i0\mathcal{L}(q))^{n-m}}\,\rho_m^{\pm}(q)\,,\qquad n=1,2,\ldots. \tag{3.32}$$

Only the leading coefficients are independent on the sign of $\mathbf{k}_{\Im}$. The coefficients with $n = 1$

$$\nu_1^{\pm}(p) = -\frac{v(p)}{p_1\pm i0\mathcal{L}(p)}\,,\qquad \rho_1^{\pm}(p) = -\int d^2q\,\frac{v(p-q)\,v(q)}{q_1\pm i0\mathcal{L}(q)}\,. \tag{3.33}$$

will be of special use in the following.

3.2.2 Properties of Spectral Data

In order to study the properties of the spectral data we, first, substitute (3.6) into (3.7) getting an integral equation for $\rho(p|\mathbf{k})$

$$\rho(p|\mathbf{k}) = v(p) + \int \frac{d^2q}{\mathcal{L}(q) - 2q_1\mathbf{k}}\, v(p-q)\,\rho(q|\mathbf{k})\,. \tag{3.34}$$

Performing the limiting procedure (3.9) we have

$$\rho^{\pm}(p|k) = v(p) + \int \frac{d^2q}{\mathcal{L}(q) - 2q_1 k \mp i0q_1}\, v(p-q)\,\rho^{\pm}(q|k) \tag{3.35}$$

and, then, recalling definition (3.11), the following representation for the spectral data

$$\rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = v(p) + p_1 \int \frac{d^2q}{p_1\mathcal{L}(q) - q_1\mathcal{L}(p) \mp i0p_1q_1}\, v(p-q)\,\rho^{\pm}\left(q\Big|\frac{\mathcal{L}(p)}{2p_1}\right)\,. \tag{3.36}$$

We deduce that for $v(p)$ vanishing rapidly at large p also the spectral data $\rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right)$ vanish rapidly at large p. However, the spectral data are discontinuous at $p = 0$. In fact if the limit $p \to 0$ is taken along the line $p_2 = 2\beta p_1$, with β an arbitrary constant, we get from (3.36)

$$\lim_{\substack{p\to 0\\ p_2 = 2\beta p_1}} \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = \rho^{\pm}(0|\beta) = v(0) + \int \frac{d^2q\,\overline{v(q)}\,\rho^{\pm}(q|\beta)}{\mathcal{L}(q) - 2\beta q_1 \mp i0q_1}\,, \tag{3.37}$$

i.e. the limit depends on β. In addition if we fix $p_2 \neq 0$ we get

$$\lim_{p_1\to 0} \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = v(0, p_2) \tag{3.38}$$

and combining the results (3.37) and (3.38)

$$\lim_{p_2\to 0}\lim_{p_1\to 0} \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right) = \lim_{\beta\to\infty}\lim_{p_1\to 0} \rho^{\pm}\left(p_1, 2\beta p_1\Big|\beta - \frac{p_1}{2}\right) = v(0)\,. \tag{3.39}$$

Note that from (3.38) it results that the spectral data $\rho^{\pm}$ at $p_1 = 0$ are smooth (Schwartz) functions of p_2. Moreover, since

$$r^{\pm}(\beta, \beta) = -2\pi i \rho^{\pm}(0|\beta) \tag{3.40}$$

and

$$\lim_{\beta\to\infty} \frac{i}{2\pi} r^{\pm}\left(\beta - \frac{p_2}{4\beta}, \beta + \frac{p_2}{4\beta}\right) = \lim_{p_1\to 0} \rho^{\pm}\left(p\Big|\frac{\mathcal{L}(p)}{2p_1}\right)\,, \tag{3.41}$$

we deduce from (3.37) and (3.38) that (no matter how rapidly decreasing at large p is chosen $v(p)$) the spectral data $r^{\pm}(\alpha, \beta)$ do not vanish, in general, at large distances in the $\{\alpha, \beta\}$ plane. The spectral data $r^{\pm}$ vanish at large distances only if the constraint

$$v(0, p_2) = 0 \tag{3.42}$$

is satisfied.

3.2.3 1/k–expansion of the Jost solution for $t \neq 0$

Since the properties of the Jost solution depend on the specific form of the spectral equation (3.4) we would expect that the introduction of a parametric dependence on t in $v(p)$ will not change the asymptotic $1/\mathbf{k}$–expansion of $\nu(p|\mathbf{k})$ or, more precisely, we would expect that the only difference would be a parametric dependence on t of the coefficients of the expansion. However, we will see that the time evolution of a solution $v(t,p)$ of the KPI equation has special properties in the neighborhoods of $t=0$ and we have to take into account that not $v(t,p)$ but $\widetilde{v}(t,p)$ as defined in the following equation

$$v(t,p) = e^{-it\frac{p_1^4+3p_2^2}{p_1}}\,\widetilde{v}(t,p) \tag{3.43}$$

is continuous at $p=0$ for $t \neq 0$.

Therefore in writing the integral equation (3.6) for the Jost solution it is convenient to factorize explicitly the singular behaviour in time as follows

$$\nu(t,p|\mathbf{k}) = \delta^2(p) + \frac{1}{\mathcal{L}(p)-2p_1\mathbf{k}}\int d^2q\, e^{-it\frac{q_1^4+3q_2^2}{q_1}}\,\widetilde{v}(t,q)\,\nu(t,p-q|\mathbf{k})\,. \tag{3.44}$$

We get

$$\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big] = \widetilde{v}(t,p)\,\lim_{\mathbf{k}\to\infty}\frac{\mathbf{k}}{\mathcal{L}(p)-2p_1\mathbf{k}}\,e^{-it\frac{p_1^4+3p_2^2}{p_1}}\,, \tag{3.45}$$

and using proposition A.1 in the appendix

$$\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big] = \frac{-1}{2(p_1+i0t)}\,e^{-it\frac{p_1^4+3p_2^2}{p_1}}\widetilde{v}(t,p) = \frac{-1}{2}\,\frac{v(t,p)}{p_1+i0t}\,. \tag{3.46}$$

Comparing this result with (3.33) we see that the coefficients of $1/\mathbf{k}$ in the asymptotic expansions coincide only if

$$\operatorname{sgn}\mathbf{k}_{\Im} = \operatorname{sgn}(t\mathcal{L}(p))\,, \tag{3.47}$$

or in other words only under this condition the limits $\mathbf{k}\to\infty$ and $t\to 0$ commute. Notice that this condition is formulated in terms of the variables p corresponding to the fact that the special behaviour of solutions and Jost solutions in the neighborhoods of $t=0$ are manageable only in the transformed Fourier space.

3.3 Inverse problem and time evolution

3.3.1 Formulation of the inverse problem

By working in the general framework of the resolvent approach [14, 8] we obtained in [15] the following coupled system of two linear integral equations for the boundary values of the Jost solutions on the real k–axis

$$\nu^{\pm}(p|k) = \delta^2(p) + \frac{1}{2}\int\frac{d^2q}{\mathcal{L}(-q)+2kq_1\pm iq_10}\sum_{\sigma=+,-}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,. \tag{3.48}$$

This formula is the main tool to be used for solving the inverse problem, i.e. for reconstructing the potential $v(p)$ starting from the spectral data and for proving that spectral data evolving in time as indicated in (3.15) generate potentials $v(t,p)$ that are solutions of the KPI equation.

In this respect two properties of this system are crucial. First, the system is closed in the sense that the spectral data $\rho^{\pm}$ are reductions of the Jost solutions themselves via equations (3.8), (3.9) and (3.11). Second, it depends linearly on the spectral data, in contrast with the usual equations written in the literature that are quadratic in the spectral data. To our knowledge the only alternative integral equations for solving the inverse problem, which are linear in the spectral data, are given in [12], but they are not closed in the sense that they are written in terms of the so called advanced/retarded solutions.

We suppose that the system (3.48) has unique solution in the considered class of spectral data. Then the r.h.s. of (3.48) can be continued analytically in the complex $\mathbf{k}$-plane as follows

$$\nu(p|\mathbf{k}) = \delta^2(p) + \frac{1}{2}\int \frac{d^2q}{\mathcal{L}(-q)+2\mathbf{k}q_1} \sum_{\sigma=\pm} \overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\, \nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big). \tag{3.49}$$

For getting from (3.49) an integral equation for $\rho(p,\mathbf{k})$ it is convenient to use equation (3.6) computed on the real axis and to rewrite the Jost solution in the r.h.s. of (3.49) as

$$\nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big) = \delta^2(q+p) + \frac{q_1\,\rho^{\sigma}\big(p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)}{q_1\mathcal{L}(p)+p_1\mathcal{L}(-q)-i\sigma 0 q_1(q_1+p_1)}. \tag{3.50}$$

Then it is easy to check that

$$\frac{\nu^{\sigma}\big(p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)}{\mathcal{L}(-q)+2\mathbf{k}q_1} = \frac{1}{\mathcal{L}(p)-2\mathbf{k}p_1}\left[-\frac{p_1}{q_1}\,\nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big) + \frac{\rho^{\sigma}\big(p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)}{\mathcal{L}(-q)+2\mathbf{k}q_1}\right], \tag{3.51}$$

where the ratio p_1/q_1 multiplying ν in the r.h.s. is well defined due to (3.50). By inserting this expression in (3.49) and by recalling definition (3.8) we get the following integral equation for $\rho(p|\mathbf{k})$

$$\begin{aligned}\rho(p|\mathbf{k}) = &-\int d^2q\,\frac{p_1}{2q_1} \sum_{\sigma=\pm} \overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\, \nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big) + \\ &\frac{1}{2}\int \frac{d^2q}{\mathcal{L}(-q)+2\mathbf{k}q_1} \sum_{\sigma=\pm} \overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\, \rho^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big).\end{aligned} \tag{3.52}$$

It has a kernel similar to the kernel of the integral equation (3.6), (3.7) obtained for $\nu(p|\mathbf{k})$ in the direct problem. Therefore its asymptotic expansion at large $\mathbf{k}$ can be computed by using the same methods used in the previous section. In particular we get that the first term in the r.h.s. due to (3.30) and (3.33) is just the potential

$$v(p) = -\int d^2q\,\frac{p_1}{2q_1}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big), \tag{3.53}$$

that is therefore explicitly reconstructed in terms of the spectral data and Jost solutions.

It is instructive to consider in details the properties of the function $v(p)$ defined by (3.53) in the neighborhoods of $p = 0$. This analysis throws some light on the meaning of the characterization equations for spectral data reported in section 3.2. In fact, if we rewrite (3.53) by using (3.50) as

$$\begin{aligned} v(p) = {} & \frac{1}{2}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(-p\Big|\frac{\mathcal{L}(-p)}{-2p_1}\Big) - \\ & \frac{p_1}{2}\int d^2q\sum_{\sigma=\pm}\frac{\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\rho^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)}{q_1\mathcal{L}(p)+p_1\mathcal{L}(-q)-i\sigma 0 q_1(q_1+p_1)}. \end{aligned} \tag{3.54}$$

and perform the same limit as in (3.37) we get

$$\begin{aligned} \lim_{p_1\to\pm 0} v(p_1, 2\beta p_1) = {} & \frac{1}{2}\sum_{\sigma=\pm}\Bigg\{\overline{\rho^{-\sigma}}(0|\beta) - \\ & \int d^2q\,\frac{\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\rho^{\sigma}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)}{\mathcal{L}(-q)+2\beta q_1 \mp i\sigma 0}\Bigg\}. \end{aligned} \tag{3.55}$$

This result seems to contradict the fact that the potential $v(p)$ was chosen to be continuous at $p = 0$. However, the term in the r.h.s which depends on the sign of p_1 is proportional to $\sum_{\sigma=\pm}\sigma\int d^2q\,\delta(\mathcal{L}(-q)+2\beta q_1)\big(\overline{\rho^{-\sigma}}\rho^{\sigma}\big)\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)$ and, then, due to (3.13) and (3.17) to $\sum_{\sigma=\pm}\sigma\int d\alpha\overline{r}^{\,-\sigma}(\alpha,\beta)r^{\sigma}(\alpha,\beta) = \sum_{\sigma=\pm}\sigma\Big(\big(\boldsymbol{\mathcal{R}}_+^{-\sigma\,\dagger} - \boldsymbol{\mathcal{R}}_-^{-\sigma\,\dagger}\big)\big(\boldsymbol{\mathcal{R}}_+^{\sigma} - \boldsymbol{\mathcal{R}}_-^{\sigma}\big)\Big)(\beta,\beta)$. But due to (3.20) and (3.21) $\big(\boldsymbol{\mathcal{R}}_+^{-\sigma\,\dagger} - \boldsymbol{\mathcal{R}}_-^{-\sigma\,\dagger}\big)\big(\boldsymbol{\mathcal{R}}_+^{\sigma} - \boldsymbol{\mathcal{R}}_-^{\sigma}\big) = 2\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^{\dagger}$ which is σ-independent. Therefore this term is equal to zero and the limit does not depend on the sign of p_1. The independence on β can be proved by using (3.37).

From (3.49) by using the expansion (3.29) we can get the coefficients of the asymptotic expansion at large $\mathbf{k}$ of $\nu(p|\mathbf{k})$. Of special interest is the coefficients of $1/\mathbf{k}$. Recalling (3.33) we have

$$\frac{v(p)}{p_1 \pm i0\mathcal{L}(p)} = -\int\frac{d^2q}{2(q_1 \mp i0\mathcal{L}(-q))}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big). \tag{3.56}$$

For the all set of coefficients of the asymptotic expansion of $\rho(p|\mathbf{k})$ we get

$$\rho_n^{\pm}(p) = \int\frac{d^2q\,(-\mathcal{L}(-q))^n}{2(q_1 \mp i0\mathcal{L}(-q))^{n+1}}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\rho^{\sigma}\Big(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big). \tag{3.57}$$

3.3.2 Jost solutions for $t \neq 0$

We choose spectral data to evolve as in (3.15)

$$\rho^{\pm}\Big(t,p\Big|\frac{\mathcal{L}(p)}{2p_1}\Big) = \exp\Big(-it\,\frac{p_1^4+3p_2^2}{p_1}\Big)\ \rho^{\pm}\Big(p\Big|\frac{\mathcal{L}(p)}{2p_1}\Big), \tag{3.58}$$

where $\rho^{\pm}(p|\frac{\mathcal{L}(p)}{2p_1})$ denotes the initial value of spectral data determined in terms of $v(p)$ and Jost solutions $\nu^{\pm}(p|k)$ according to formula (3.11).

Then the Jost solution at any time is reconstructed just by inserting these spectral data into (3.49)

$$\nu(t,p|\mathbf{k}) = \delta^2(p) + \frac{1}{2}\int \frac{d^2q\, e^{it\frac{q_1^4+3q_2^2}{q_1}}}{\mathcal{L}(-q)+2\mathbf{k}q_1}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big). \tag{3.59}$$

In the limit $\mathbf{k} \to k \pm i0$ (cf. (3.48)) we obtain the integral equations for the boundary values on the real axis, i.e. for the functions $\nu^{\pm}(t,p|k)$. In what follows we always suppose that these equations are uniquely solvable.

We have, first, to prove that this function $\nu(t,p|\mathbf{k})$ is indeed the Fourier transformed of the Jost solution of (1.2) , i.e. that it satisfies (3.4), and, second, we have to find its time evolution equation, which is needed to know in order to show that the reconstructed $v(t,p)$ satisfies KPI.

3.3.3 The recontructed spectral equation

First we prove that ν defined by (3.59) obeys the spectral equation (3.4) at any time. Let us define $\rho(t,p|\mathbf{k})$ by (3.8) rewritten for any time

$$\rho(t,p|\mathbf{k}) = [\mathcal{L}(p) - 2p_1\mathbf{k}]\,\nu(t,p|\mathbf{k}). \tag{3.60}$$

Then, by the same procedure used to get (3.52) from (3.49) we derive from (3.59) that

$$\rho(t,p|\mathbf{k}) = v(t,p) + \frac{1}{2}\int \frac{d^2q\, e^{it\frac{q_1^4+3q_2^2}{q_1}}}{\mathcal{L}(-q)+2\mathbf{k}q_1}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\rho^{\sigma}\Big(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,, \tag{3.61}$$

where we have introduced

$$v(t,p) = -\int d^2q\,\frac{p_1}{2q_1}\ e^{it\frac{q_1^4+3q_2^2}{q_1}}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big). \tag{3.62}$$

Now due to the assumption on unique solvability of (3.59) we have from (3.61) that $\rho(t,p|\mathbf{k}) = \int d^2q v(t,p-q)\nu(t,q|\mathbf{k})$, i.e. (3.7) is satisfied for any t. Thanks to (3.8) this proves that (3.4) is satisfied at any time.

To prove that ν defined by (3.59) obeys the integral equation (3.6) at any time we use (3.59), (3.61) and (3.62) to get

$$\begin{aligned}
&\nu(t,p|\mathbf{k}) - \delta^2(p) - \frac{\rho(t,p|\mathbf{k})}{\mathcal{L}(p) - 2p_1\mathbf{k}} = \\
&\frac{1}{2}\int \frac{d^2q\, e^{it\frac{q_1^4+3q_2^2}{q_1}}}{\mathcal{L}(-q) + 2\mathbf{k}q_1} \sum_{\sigma=\pm} \overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big) \times \\
&\left[\nu^\sigma\Big(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\frac{\mathcal{L}(p+q) - 2(p_1+q_1)\frac{\mathcal{L}(q)}{2q_1}}{\mathcal{L}(p) - 2\mathbf{k}p_1} - \frac{\rho^\sigma\big(t,p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)}{\mathcal{L}(p) - 2\mathbf{k}p_1}\right]. \qquad (3.63)
\end{aligned}$$

Then (3.6) at any t follows by using definition (3.60) of ρ^σ in the r.h.s. This proves that $\nu(t,p|\mathbf{k})$ in (3.59) is the Fourier transform of the Jost solution of the equation (1.2) with potential the Fourier transform of $v(t,p)$ defined in (3.62).

To prove that $v(t,p)$ satisfies equation (3.43) with $\widetilde{v}(t,p)$ continuous at $p=0$ it is convenient to insert (3.50) (that we proved to be satisfied at any time) into (3.62) getting

$$\begin{aligned}
v(t,p) = {}& e^{-it\frac{p_1^4+3p_2^2}{p_1}}\,\frac{1}{2}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(-p\Big|\frac{\mathcal{L}(-p)}{-2p_1}\Big) - \\
&\frac{1}{2}\int d^2q\, p_1 e^{it\frac{q_1^4+3q_2^2}{q_1}} \sum_{\sigma=\pm}\frac{\overline{\rho^{-\sigma}}\big(q\big|\frac{\mathcal{L}(q)}{2q_1}\big)\;\rho^\sigma\big(t,p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)}{q_1\mathcal{L}(p) + p_1\mathcal{L}(-q) - i\sigma 0 q_1(q_1+p_1)}. \qquad (3.64)
\end{aligned}$$

The time evolution of $\rho^\sigma\big(t,p+q\big|\frac{\mathcal{L}(q)}{2q_1}\big)$ in the r.h.s. is not explicitly known. However, it is useful to factorize the time evolution it would have in the limit $v(p)\to 0$ by writing (see also (3.81) below)

$$\rho(t,p|\mathbf{k}) = e^{-2it[3(\mathbf{k}+p_1)\mathcal{L}(p)+2p_1^3]}\,\widetilde{\rho}(t,p|\mathbf{k}). \qquad (3.65)$$

Note that for the special reduction furnishing the spectral data we have that

$$\widetilde{\rho}^\sigma\left(t,q\Big|\frac{\mathcal{L}(q)}{2q_1}\right) = \rho^\sigma\left(q\Big|\frac{\mathcal{L}(q)}{2q_1}\right) \qquad (3.66)$$

is time independent.

Finally we can write by using the identity

$$\begin{aligned}
\frac{q_1^4+3q_2^2}{q_1} = {}& 6\mathcal{L}(p)\frac{\mathcal{L}(-q)}{2q_1} - 2[3p_1\mathcal{L}(p) + 2p_1^3] + \\
&2\left[3\left(\frac{\mathcal{L}(q)}{2q_1} + q_1 + p_1\right)\mathcal{L}(p+q) + 2(p_1+q_1)^3\right] \qquad (3.67)
\end{aligned}$$

that

$$\widetilde{v}(t,p)-v(p)=-\frac{1}{2}\int d^2q\,p_1\sum_{\sigma=\pm}\frac{\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)}{q_1\mathcal{L}(p)+p_1\mathcal{L}(-q)-i\sigma 0q_1(q_1+p_1)}\times \tag{3.68}$$
$$\left[\exp\left(6it\mathcal{L}(p)\left(\frac{\mathcal{L}(p)}{2p_1}+\frac{\mathcal{L}(-q)}{2q_1}\right)\right)\widetilde{\rho}^{\sigma}\left(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)-\rho^{\sigma}\left(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)\right].$$

At the special values of p and q for which the denominator in the r.h.s. vanishes

$$p+q\Big|_{q_1\mathcal{L}(p)+p_1\mathcal{L}(-q)=0}=\left\{\beta-\alpha,\beta^2-\alpha^2\right\}, \tag{3.69}$$

with

$$\alpha=\frac{\mathcal{L}(q)}{2q_1}, \qquad \beta=\frac{\mathcal{L}(-p)}{-2p_1} \tag{3.70}$$

and we obtain that, due to (3.13) and (3.66), $\widetilde{\rho}$ and ρ reduce to the same spectral data

$$\rho^{\sigma}(\beta-\alpha,\beta^2-\alpha^2|\alpha)\,. \tag{3.71}$$

Therefore, at $q_1\mathcal{L}(p)+p_1\mathcal{L}(-q)=0$ the function in the square brackets in the r.h.s. vanishes and p_1 can be extracted from the integral. We conclude that

$$\lim_{p\to 0}\widetilde{v}(t,p)=v(0). \tag{3.72}$$

The continuity of $\widetilde{v}(t,p)$ at $p=0$ ensures us that the exponent in (3.43) is the dominant term in the time evolution of $v(t,p)$ and therefore we recover for $u(t,x,y)$ at large x the behaviour computed in the linearized case in (2.104).

3.3.4 1/k–expansion in the inverse problem

One can compute the coefficient of $1/\mathbf{k}$ in the asymptotic expansion at large $\mathbf{k}$ of $\nu(t,p|\mathbf{k})$ also by using the Jost solution as reconstructed from spectral data via the integral equation (3.59).

Thanks to proposition A.1 in the appendix we have that

$$\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big]=$$
$$\frac{1}{4}\int\frac{d^2q\,e^{it\frac{q_1^4+3q_2^2}{q_1}}}{q_1-i0t}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\Big(q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,\nu^{\sigma}\Big(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\Big)\,. \tag{3.73}$$

As mentioned in the appendix the $i0t$ term is not relevant for $t\neq 0$ and, therefore, we get from (3.62)

$$\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big]=-\frac{v(t,p)}{2p_1}\qquad \text{for } t\neq 0\,, \tag{3.74}$$

The singular function $1/p_1$ does not need any regularization at $t \neq 0$ since it is smoothed by the oscillating behaviour in time of $v(t,p)$ indicated in (3.43). On the other side at $t = 0$ (3.73) is discontinuous

$$\lim_{t\to\epsilon 0}\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big]=$$
$$\frac{1}{4}\sum_{\sigma=\pm}\int\frac{d^2q}{q_1-i\epsilon 0}\,\overline{\rho^{-\sigma}}\left(q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)\,\nu^{\sigma}\left(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\right),\qquad \epsilon=+,-\,. \tag{3.75}$$

Substituting $(q_1 - i\epsilon 0)^{-1} = (q_1 \mp i\mathcal{L}(-q)0)^{-1} + 2i\pi\epsilon\delta(q_1)\vartheta(\pm\epsilon q_2)$ we can use (3.56) and obtain

$$\lim_{t\to\epsilon 0}\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big]=-\frac{v(p)}{2(p_1\pm ip_2 0)}+$$
$$\frac{i\pi\epsilon}{2}\sum_{\sigma=\pm}\int dq_2\lim_{q_1\to 0}\,\overline{\rho^{-\sigma}}\left(q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)\,\nu^{\sigma}\left(p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)\vartheta(\pm\epsilon q_2)\,,\qquad \epsilon=+,-\,. \tag{3.76}$$

Due to (3.38) and (3.50) the last term is equal to $i\pi\epsilon\vartheta(\mp\epsilon p_2)\delta(p_1)v(0,p_2)$ and, consequently,

$$\lim_{t\to\epsilon 0}\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\nu(t,p|\mathbf{k})-\delta^2(p)\big]=-\frac{v(p)}{2(p_1+i\epsilon 0)}\,, \tag{3.77}$$

Combining this result with the previous formula (3.74) we recover the same coefficient (3.46) computed by using the integral equation for $\nu(t,p|\mathbf{k})$ of the direct problem and, by recalling (3.73) and (3.74), we obtain that

$$\frac{v(t,p)}{p_1+i0t}=-\frac{1}{2}\int\frac{d^2q\,e^{it\frac{q_1^4+3q_2^2}{q_1}}}{q_1-i0t}\sum_{\sigma=\pm}\overline{\rho^{-\sigma}}\left(q\Big|\frac{\mathcal{L}(q)}{2q_1}\right)\nu^{\sigma}\left(t,p+q\Big|\frac{\mathcal{L}(q)}{2q_1}\right). \tag{3.78}$$

3.3.5 The recontructed time evolution of the Jost solution

To derive the time evolution equation of the Jost solution we differentiate with respect to t equation (3.59). Proposition A.1 in the appendix furnishes the derivative of the kernel of this equation

$$\frac{\partial}{\partial t}\frac{e^{it\frac{q_1^4+3q_2^2}{q_1}}}{\mathcal{L}(-q)+2\mathbf{k}q_1}=3i\mathcal{L}(-q)\frac{e^{it\frac{q_1^4+3q_2^2}{q_1}}+}{q_1-i0t}+$$
$$2i(2q_1^3+3(q_1-\mathbf{k})\mathcal{L}(-q))\,\frac{e^{it\frac{q_1^4+3q_2^2}{q_1}}}{\mathcal{L}(-q)+2\mathbf{k}q_1}\,. \tag{3.79}$$

By using (3.78) and the identity

$$\frac{q_1^4+3q_2^2}{q_1}=6\mathcal{L}(p)\left(\mathbf{k}+\frac{\mathcal{L}(-q)}{2q_1}\right)-2[3(\mathbf{k}+p_1)\,\mathcal{L}(p)+2p_1^3]+$$

$$2\left[3\left(\frac{\mathcal{L}(q)}{2q_1}+q_1+p_1\right)\mathcal{L}(p+q)+2(p_1+q_1)^3\right] \tag{3.80}$$

we obtain an integral equation of the type (3.59) for the quantity $\partial\nu(t,p|\mathbf{k})/\partial t+2i\,\nu(t,p|\mathbf{k})$ $\left[3(\mathbf{k}+p_1)\,\mathcal{L}(p)+2p_1^3\right]$ with the inhomogeneous term equal to $-3iv(t,p)\,\frac{\mathcal{L}(p)}{p_1+i0t}$. Thanks to the assumption of uniqueness of the solution of this equation we get, finally, the searched time evolution equation for the Jost solution

$$\frac{\partial\nu(t,p|\mathbf{k})}{\partial t}=-2i\,\nu(t,p|\mathbf{k})\big[3(\mathbf{k}+p_1)\,\mathcal{L}(p)+2p_1^3\big]-3i\int d^2q\,v(t,q)\,\frac{\mathcal{L}(q)}{q_1+i0t}\,\nu(t,p-q|\mathbf{k})\,. \tag{3.81}$$

As expected the time derivative of $\nu(p|\mathbf{k})$ is discontinuous at $t=0$

$$\lim_{t\to\pm0}\frac{\partial\nu(t,p|\mathbf{k})}{\partial t}=-2i\,\nu(p|\mathbf{k})\big[3(\mathbf{k}+p_1)\,\mathcal{L}(p)+2p_1^3\big]-3i\int d^2q\,v(q)\,\frac{\mathcal{L}(q)}{q_1\pm i0}\,\nu(p-q|\mathbf{k})\,. \tag{3.82}$$

Note that at large $\mathbf{k}$ the leading term in the l.h.s. is of order $1/\mathbf{k}$ while in the r.h.s. it is of order $\mathbf{k}^0$ and, due to (3.33), it is equal to

$$3iv(p)\left(\frac{\mathcal{L}(p)}{p_1+i\sigma\mathcal{L}(p)0}-\frac{\mathcal{L}(p)}{p_1\pm i0}\right)\,,\qquad \sigma=\operatorname{sgn}\mathbf{k}_{\Im}\,, \tag{3.83}$$

This term, if the initial data $v(p)$ do not satisfy the constraint $v(0,p_2)=0$, is zero only if $\operatorname{sgn}\mathbf{k}_{\Im}=\operatorname{sgn}(t\mathcal{L}(p))$. Once more we recover the relevant fact that the two limits $t\to 0$ and $\mathbf{k}\to\infty$ can be exchanged only if condition (3.47) is satisfied.

3.3.6 The reconstructed KPI equation and the integrals of motion

Recalling (3.8), that we proved to be satisfied at all times, we deduce from (3.81) that

$$\begin{aligned}\frac{\partial\rho(t,p|\mathbf{k})}{\partial t}&=-6i\,\mathcal{L}(p)\mathbf{k}\left[\rho(t,p|\mathbf{k})-v(t,p)\right]-\\ 3i\frac{\mathcal{L}(p)^2}{p_1+i0t}\,&v(t,p)-2ip_1\big[3\mathcal{L}(p)+2p_1^2\big]\,\rho(t,p|\mathbf{k})-\\ 3i[\mathcal{L}(p)-2p_1\mathbf{k}]&\int d^2q\,v(t,q)\,\frac{\mathcal{L}(q)}{q_1+i0t}\left[\nu(t,p-q|\mathbf{k})-\delta^2(p-q)\right].\end{aligned} \tag{3.84}$$

The time evolution equation of $v(t,p)$ can be obtained from this equation by computing the coefficient of the $1/\mathbf{k}$ term in its asymptotic expansion at large $\mathbf{k}$.

We first note that from (3.46), by using (3.7) that we proved to be satisfied at all times, we get

$$\lim_{\mathbf{k}\to\infty}\mathbf{k}\big[\rho(t,p|\mathbf{k})-v(t,p)\big]=-\int d^2q\frac{v(t,p-q)\,v(t,q)}{2(q_1+i0t)}\,. \tag{3.85}$$

Then we multiply (3.84) by $\mathbf{k}$ and compute the limit for $\mathbf{k} \to \infty$ by inserting (3.46) and (3.85). We obtain

$$\begin{aligned}
&\frac{\partial v(t,p)}{\partial t} = 3i\,\mathcal{L}(p) \int d^2q \frac{v(t,p-q)\,v(t,q)}{q_1 + i0t} - \\
&i \left(\frac{3\mathcal{L}(p)^2}{p_1 + i0t} + 6p_1 \mathcal{L}(p) + 4p_1^3 \right) v(t,p) - \\
&3ip_1 \int d^2q\, v(t,q)\, \frac{\mathcal{L}(q)}{q_1 + i0t}\, \frac{v(t,p-q)}{p_1 - q_1 + i0t}\,,
\end{aligned} \tag{3.86}$$

which can be easily transformed into the evolution form of the KPI equation (1.15) reported in the introduction.

Of special interest is the behaviour of equation (3.84) in a neighborhood of the point of discontinuity $p = 0$. Precisely we consider the limit $p \to 0$ along a path such that $\lim_{p\to 0} p_2^2/p_1 = 0$. Note that, because of (3.43) and the properties of $\widetilde{v}(t,p)$ at $p = 0$, only in this limit $v(t,0) = v(0)$. Then all the terms in the r.h.s. go to zero and we get that

$$\lim_{\substack{p\to 0\\ p_2^2/p_1 \to 0}} \frac{\partial \rho(t,0|\mathbf{k})}{\partial t} = 0. \tag{3.87}$$

From (3.61), recalling (3.58), we get the dispersion relation

$$C(\mathbf{k}) \equiv \lim_{\substack{p\to 0\\ p_2^2/p_1 \to 0}} \rho(t,0|\mathbf{k}) = v(0) + \frac{1}{2} \int \frac{d^2q}{\mathcal{L}(-q) + 2\mathbf{k}q_1} \sum_{\sigma=\pm} \overline{\rho^{-\sigma}} \Big(q \Big| \frac{\mathcal{L}(q)}{2q_1} \Big)\, \rho^{\sigma} \Big(q \Big| \frac{\mathcal{L}(q)}{2q_1} \Big)\,. \tag{3.88}$$

We conclude that $C(\mathbf{k})$ generate an infinite number of conserved quantities, that can be obtained by the recursion relations (3.32) computed at $p = 0$, when the limit is taken in the specified way. These conserved quantities are expressed in terms of the spectral data by means of equations (3.57) evaluated at $p = 0$.

4 Concluding Remarks

We conclude by attracting the attention of the reader to some specific features of the formulation of the direct and inverse problems considered in this article and in [13].

1. The direct and inverse problems are formulated in terms of the Fourier transformed Jost solutions. It results that the corresponding variables p_1 and p_2 are more proper for the study of the properties of these solutions than the original ones x and y. For instance condition (3.47) which play a relevant role in the theory cannot be reformulated in terms of the x, y variables.

2. We use variables p_1 and p_2 and their combinations as arguments of the spectral data instead of the standard variables α and β (see (3.12) and (3.13)). These variables

are suggested by the form of the direct problem. Properties of the spectral data are more naturally formulated in their terms.

3. The main tool of the inverse problem is a set of two integral equations for the boundary values of the Jost solutions on the real axis of the complex plane of the spectral parameter $\mathbf{k}$. This set depends linearly and not quadratically, as in the standard formulation, on the spectral data. In addition the spectral data themselves are given in terms of the Jost solutions by means of (3.11). This closed formulation linear in the spectral data enables us, for instance, to study easier the properties of the Jost solutions under time evolution and to deduce easier the asymptotic expansion at large $\mathbf{k}$ and dispersion relation (3.88).

Functional character of spectral data and their different properties according to the choice of spectral variables require some special remarks.

If we choose the standard set of spectral data (3.13) we get spectral data which have bad behaviour for α and β going to infinity. More precisely, due to (3.37), (3.38), (3.40) and (3.41), the diagonal values $r^{\pm}(\beta,\beta)$ and values 'close' to the diagonal do not decrease at infinity (see also [13]). It is necessary to emphasize that this happens for initial data of KPI as small in norm and smooth as we like. The only possibility to remove this constant behavior is to impose the constraint $v(0,p_2)=0$ (i.e. (1.5)) on initial data. In this case we get a $1/\beta$ behavior, to improve which we need a second constraint and so on. Due to (3.37) it is clear that in order to get spectral data $r^{\pm}(\beta,\beta)$ decreasing for $\beta\to\infty$ quicker then any negative power of β it is necessary to impose an infinite set of constraints on initial data. Let us mention that this situation is specific of the $2+1$ dimensional case and has no analogue in $1+1$.

In [13] it was shown that the study of the properties of the Jost solutions by means of the inverse problem in terms of spectral data $r^{\pm}(\alpha,\beta)$ needs a special care due to this asymptotic behavior. In order to compute the asymptotic $1/\mathbf{k}$–expansion or the time derivative of the Jost solutions it is necessary to use a special subtraction procedure to avoid divergent expressions.

The use of the spectral data $\rho^{\pm}\left(p,\frac{\mathcal{L}(p)}{2p_1}\right)$ in this article enables us to remove this complications. This functions rapidly decrease for $p\to\infty$ in all directions (roughly speaking as initial data $v(p)$) and are smooth at $p_1\neq 0$ (see (3.36)). Of course difficulties cannot disappear thanks to a convenient change of variables and in fact also the spectral data $\rho^{\pm}\left(p,\frac{\mathcal{L}(p)}{2p_1}\right)$ do not belong to the Schwartz space since they are discontinuous at $p=0$. This difficulty is, however, more naturally manageable and direct and inverse problems can be formulated practically in the standard way. Special attention must be paid to the discontinuity of the time derivatives at $t=0$. This problem is controlled by the lemma and propositions given in the appendix.

As shown in section 3 the properties of all the quantities under consideration are essentially different for $t=0$ and $t\neq 0$. This situation is specific of the multi–dimensional integrable evolution equations, which are non local already in their linearized version. In fact the nonlocality implies that the dispersion relation of spectral data is singular as indicated for instance in (3.58). Then for $t \neq 0$ the rapid oscillations of the exponent in (3.58) near $p_1 = 0$ can control a singularity at $p_1 = 0$ while for $t = 0$ a specific regularization is needed. This is hidden if we use the standard spectral data $r^{\pm}(\alpha, \beta)$. In fact the exponent in (3.14) changes the functional class of the initial data. The diagonal values $r^{\pm}(t, \beta, \beta)$ are time independent and thus have the same bad behaviour at large β as the spectral data at time $t = 0$, but asymptotic expansions 'close' to the diagonal of the type (3.41) show again rapid oscillations for t and p_2 different from zero.

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A Appendix

Lemma A.1 *The function* $\exp(i\tau/p_1)$ *defines a distribution in the Schwartz space of the variable* p_1 *depending continuously on the parameter* τ.

i) This distribution is continuously differentiable in τ *for any* $\tau \neq 0$. *Precisely, we have*

$$\partial_\tau \exp\left(\frac{i\tau}{p_1}\right) = \frac{i}{p_1} \exp\left(\frac{i\tau}{p_1}\right), \qquad \tau \neq 0, \tag{A.1}$$

where the r.h.s. is a well defined distribution in the same space.

ii) At $\tau = 0$ *there exist right/left limits*

$$\lim_{\tau\to\pm 0} \frac{1}{p_1} \exp\left(\frac{i\tau}{p_1}\right) = \frac{1}{p_1 \mp i0}. \tag{A.2}$$

Thus $\partial_\tau \exp\left(\frac{i\tau}{p_1}\right)$ *can be considered a distribution also in the variable* τ *and we can write for arbitrary* τ

$$\partial_\tau \exp\left(\frac{i\tau}{p_1}\right) = \frac{i}{p_1 - i0\tau} \exp\left(\frac{i\tau}{p_1}\right). \tag{A.3}$$

Proof

i) Let us consider

$$f(\tau) = \int dp_1\, e^{i\tau/p_1} \varphi(p_1)$$

where $\varphi(p_1) \in \mathcal{S}$. Subtracting and adding terms we can write

$$f(\tau) = \int dp_1\, e^{i\tau/p_1} \left[\varphi(p_1) - \vartheta(1 - |p_1|)\varphi(0)\right] + \varphi(0) \int_{-1}^{1} dp_1\, e^{i\tau/p_1}.$$

Changing p_1 to $1/p_1$ in the second term we have

$$f(\tau) = \int dp_1\, e^{i\tau/p_1}\big[\varphi(p_1) - \vartheta(1-|p_1|)\varphi(0)\big] + \varphi(0)\int_{|p_1|\geq 1} \frac{dp_1}{p_1^2}\, e^{i\tau p_1}.$$

Now it is obvious that both terms are differentiable in τ for $\tau \neq 0$ and

$$\partial_\tau f(\tau) = i\int \frac{dp_1}{p_1}\, e^{i\tau/p_1}\big[\varphi(p_1) - \vartheta(1-|p_1|)\varphi(0)\big] + i\varphi(0)\int_{|p_1|\geq 1} \frac{dp_1}{p_1}\, e^{i\tau p_1}.$$

Substituting again p_1 for $1/p_1$ in the second term we obtain that it cancels out with the second term in brackets. This proves *i)*. Let us remark that the last formula shows that for $\tau \neq 0$ it is not necessary to regularize the factor $1/p_1$ in front of the exponent in (A.1). As well one can consider this factor indifferently as principal value or as $(p_1 \pm i0)^{-1}$.

ii) Again subtracting and adding terms we can write

$$\begin{aligned}\int \frac{dp_1}{p_1}\, e^{i\tau/p_1}\, \varphi(p_1) = &\int \frac{dp_1}{p_1} e^{i\tau/p_1}\, [\varphi(p_1) - \vartheta(1-|p_1|)\varphi(0)] + \\ &\operatorname{sgn}\tau\, \varphi(0)\int_{|p_1|\geq|\tau|} \frac{dp_1}{p_1}\, e^{ip_1},\end{aligned}$$

where now in the second term p_1 was substituted for τp_1. To compute the limit for $\tau \to 0$ notice that

$$\lim_{\tau\to 0}\int_{|p_1|\geq|\tau|} \frac{dp_1}{p_1}\, e^{ip_1} = i\pi.$$

Thus

$$\begin{aligned}\lim_{\tau\to 0}\int \frac{dp_1}{p_1}\, e^{i\tau/p_1}\varphi(p_1) = &\int \frac{dp_1}{p_1}\, [\varphi(p_1) - \vartheta(1-|p_1|)\varphi(0)] \pm i\pi\varphi(0) \equiv \\ &\int \frac{dp_1}{p_1}\, \varphi(p_1) \pm i\pi\varphi(0),\end{aligned}$$

where to get the second equality we chose p_1^{-1} in the integral as the principal value. Then (A.3) follows due to (A.1) and the remark made after the proof of *i)*.

Some useful propositions follow from this lemma.

Proposition A.1 *The function* $\frac{e^{-3it\frac{p_2^2}{p_1}}}{p_2 - 2\mathbf{k}p_1}$ *defines a distribution in the Schwartz space of the variables* p_1, p_2 *depending continuously on the parameter* t.

i) This distribution is continuously differentiable in t *for any* $t \neq 0$ *and has well defined right/left limits at* $t = 0$ *according to the following formula*

$$\frac{\partial}{\partial t}\frac{e^{-3it\frac{p_2^2}{p_1}}}{p_2 - 2\mathbf{k}p_1} = -\frac{6i\mathbf{k}p_2}{p_2 - 2\mathbf{k}p_1} e^{-3it\frac{p_2^2}{p_1}} - \frac{3ip_2}{p_1 + i0t} e^{-3it\frac{p_2^2}{p_1}} \qquad \text{(A.4)}$$

ii)

$$\lim_{\mathbf{k}\to\infty} \frac{\mathbf{k}}{p_2 - 2\mathbf{k}p_1} e^{-3it\frac{p_2^2}{p_1}} = -\frac{1}{2}\frac{e^{-3it\frac{p_2^2}{p_1}}}{p_1 + i0t} \tag{A.5}$$

Note that, due to (3.29),

$$\lim_{\mathbf{k}\to\infty} \frac{\mathbf{k}}{p_2 - 2\mathbf{k}p_1} = -\frac{1}{2(p_1 + i0\sigma p_2)}, \qquad \sigma = \operatorname{sgn} \mathbf{k}_{\Im}, \tag{A.6}$$

and therefore we get from the previous proposition:

Proposition A.2 *The limits in the following formula*

$$\lim_{t\to\pm 0} \lim_{\mathbf{k}\to\infty} \frac{\mathbf{k}}{p_2 - 2\mathbf{k}p_1} e^{-3it\frac{p_2^2}{p_1}} \tag{A.7}$$

can be exchanged if and only if

$$\operatorname{sgn} \mathbf{k}_{\Im} = \operatorname{sgn} tp_2. \tag{A.8}$$

From the point *ii)* of lemma A.1 we get that

Proposition A.3 *For any distribution defined by*

$$U(t,x,y) = \int d^2p\, e^{-ip_1x + ip_2y} \exp\left(-it\,\frac{p_1^4 + 3p_2^2}{p_1}\right) V(p) \tag{A.9}$$

with $V(p)$ an arbitrary function belonging to the Schwartz space

$$\lim_{t\to\pm 0} \int_{\infty}^{x} dx' U(t,x',y) = \int_{\mp\infty}^{x} dx' U(0,x',y) \tag{A.10}$$

$$\lim_{t\to\pm 0} \int_{-\infty}^{x} dx' U(t,x',y) = \int_{\mp\infty}^{x} dx' U(0,x',y). \tag{A.11}$$

Therefore the operator inverse of ∂_x defined by $\int_{-t\infty}^{x} dx'$ commute with the limit $t \to 0$

$$\lim_{t\to\pm 0} \int_{-t\infty}^{x} dx' U(t,x',y) = \int_{\mp\infty}^{x} dx' U(0,x',y) \tag{A.12}$$

while the antisymmetrical inverse operator satisfies

$$\lim_{t\to\pm 0} \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) dx' U(t,x',y) = \int_{\mp\infty}^{x} dx' U(0,x',y). \tag{A.13}$$

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